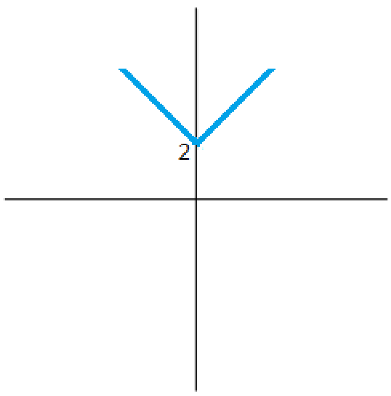
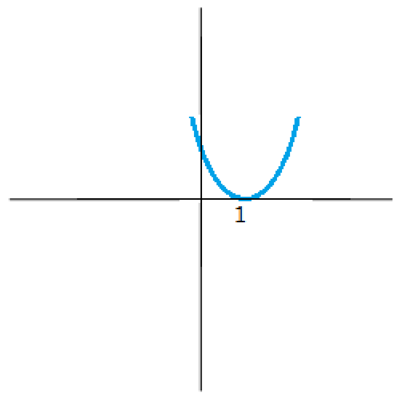
Algebra and Calculus Final

Name: Solutions

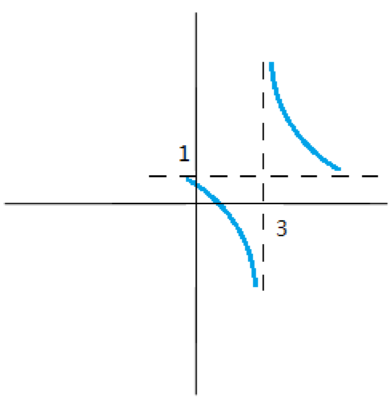
Problem 1



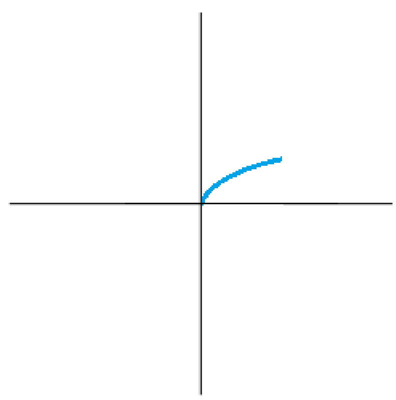
$y = |x| + 2$



$y = (x-1)^2$



$y = \frac{x-2}{x-3}$



$y = \sqrt{x}$

a. Sketch $y = |x| + 2$. First draw

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

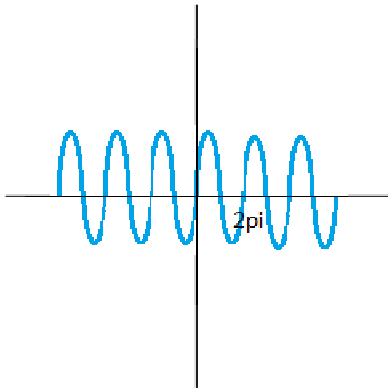
so $|x|$ is a piecewise function, then draw $|x| + 2$ by moving the plot $|x|$ up by 2 units. Recall graph translation: suppose $a > 0$, changing x to $x - a$ means to shift the plot to $x+$ direction by a units; while changing $x \to x + a$ means to shift the plot to $x-$ direction by a units. Similarly changing y to $y-a$ means to shift the plot to $y+$ direction by a units; while changing $y \to y+a$ means to shift the plot to $y-$ direction by a units. Here we have $y-2=|x|$ so we move $y=|x|$ in $y+$ direction by 2 units.

b. Sketch $y=(x-1)^2$. First draw $y=x^2$, then move the graph to the left by 1 unit.

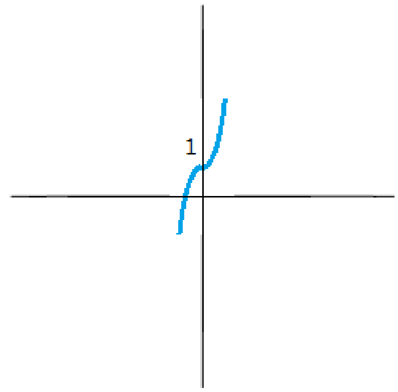
c. Sketch $y=\frac{x-2}{x-3}$. First write $\frac{x-2}{x-3}=1+\frac{1}{x-3}$, so draw $1/x$ then move the graph to the left by 3 units and then up by 1 unit.

d. Sketch $y=\sqrt{x}$. First recognize that the domain of the function is $x \geq 0$. Draw $y=x^2$ then exchange the roles of x and y, so $y=x^2$ becomes $x=y^2$. To do so, we can relabel the original x axis to y axis and relabel the original y axis to x axis, or more conventionally instead of relabeling axes, we flip the graph about the 45^0 line, i.e. $y=x$ line. Now we have the graph of $x=y^2$, which is identical to $\sqrt{x}=y$ if we restrict y to be non negative, hence we should erase the part of $y<0$ portion of $x=y^2$, so we get the desired graph.

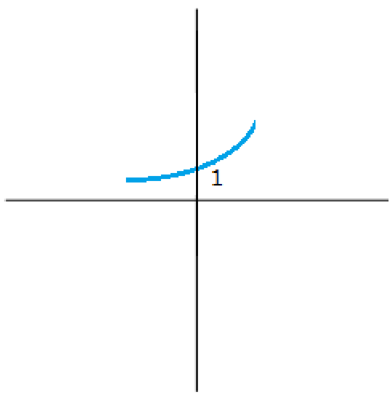
Problem 2



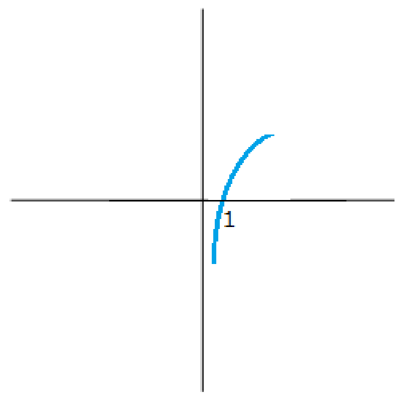
$y = \sin x$



$y = x^3 + 1$



$y = e^x$



$y = \ln x$

a. This is sinusoidal curve. Remember $\sin 0 = 0$ so it passes through the origin and the period, T, is defined to the smallest positive number such that $\sin x = \sin(x + T)$, so $T = 2\pi$.

b. First draw x^3 and remember the differences between x^3 and x^2: x^3 increases faster than x^2 so x^3 is steeper than x^2. And x^3 is odd function while x^2 is even.

c. exponential function grows like a "exponential", i.e. faster than any polynomials. When $x < 0$, think e^x as

$$\frac{1}{e^{-x}}$$

so e^{-x} still grows to infinity but 1/infinity goes to 0, so for $x < 0$ the curve is bounded by the x axis.

d. Do the inversion, i.e. flip the graph of e^x about the 45^0 line. (cf problem 1(d) above)

Problem 3

a. Complete the square

$$f = -2(x-3)^2 + 11$$

Graphically f is a concave down parabola with the top point at $(3, 11)$ so max value of f is 11. Or algebraically since $-2(x-3)^2$ is always non positive for any value of x. Adding something non positive always makes things smaller, hence the maximum value is 11.

b. We have

$$3g + 1 = \ln(x+3)$$

exponentiate both sides

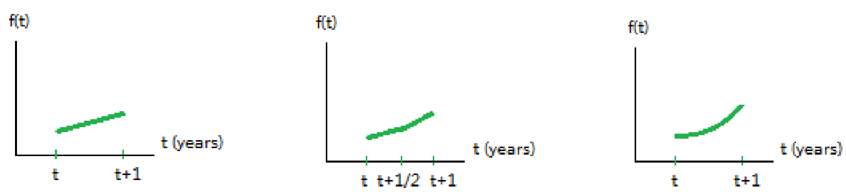
$$e^{3g+1} = x + 3$$

Hence

$$x = e^{3g+1} - 3$$

The function is invertible because the original function is one to one and onto.

c. Let $f(t)$ be the function of balance of the account in t years after initial deposit. So in our case the base year $f(0) = 10000$.



Compounded Annually
Compounded Bi-annually
Compounded Continuously

If it is compounded annually, we know that the rate of annual growth

$$\frac{f(t+1) - f(t)}{1} = 0.03f(t)$$

or equivalently

$$f(t+1) = f(t) + 0.03f(t) = 1.03f(t)$$

where $0.03f(t)$ is the interest being calculated and added to the principal in one year.

So by recursion and trace back to the base year, we have

$$f_{annual}(t) = (1.03)^t f(0)$$

If it is compounded bi-annually, we know that the bank will calculate and add internet to the principal every half year, so

$$\begin{cases} f(t+\frac{1}{2}) = f(t) + \frac{0.03}{2}f(t) = (1+\frac{0.03}{2})f(t) \\ f(t+1) = (1+\frac{0.03}{2})f(t+\frac{1}{2}) \end{cases}$$

Hence

$$f(t+1) = (1+\frac{0.03}{2})^2 f(t)$$

- Note that if it were non-compounding (called simple interest), then at the half year (with bi-annual interest rate 0.03/2) interest

  $$\frac{0.03}{2}f(t)$$

  is paid but not added to the principal, and at the end year another (with bi-annual interest rate 0.03/2) interest

  $$\frac{0.03}{2}f(t)$$

  is paid, so after all total annual interest is

  $$\frac{0.03}{2}f(t) + \frac{0.03}{2}f(t) = 0.03f(t)$$

  which is the same amount as if interest is paid annually. Because of this consistency requirement, bi-annual interest rate is 1/2 of the annual interest rate. The same logic applies to quarter rate, monthly rate, etc.

- Note that one can also see that the effective annual growth rate for compounded bi-annually is

  $$\frac{f(t+1)-f(t)}{1} = [(1+\frac{0.03}{2})^2 - 1]f(t) = [0.03 + (\frac{0.03}{2})^2]f(t)$$

  Comparing this to the annual growth rate of compounded annually, there is an addition interest $(\frac{0.03}{2})^2 f(t)$, due to the additional interest added to the principal at the half year.

Similarly by recursion, we get

$$f_{bi-annual}(t) = [(1+\frac{0.03}{2})^2]^t f(0)$$

If it is compounded monthly, i.e. 12 times a year

$$\begin{cases} f(t+\frac{1}{12}) = f(t) + \frac{0.03}{12} f(t) = (1+\frac{0.03}{12}) f(t) \\ f(t+\frac{2}{12}) = (1+\frac{0.03}{12}) f(t+\frac{1}{12}) \\ \dots \\ f(t+1) = (1+\frac{0.03}{12}) f(t+\frac{11}{12}) \end{cases}$$

That is

$$f(t+1) = (1+\frac{0.03}{12})^{12} f(t)$$

so

$$f_{monthly}(t) = (1+\frac{0.03}{12})^{12t} f(0)$$

Therefore it is easy to see the pattern that if the bank calculates and adds interest to the balance N times a year (N is any arbitrary number), we claim that

$$f(t+1) = (1+\frac{0.03}{N})^{N} f(t)$$

Or

$$f_{componuded\, N\, times\, per\, yr}(t) = (1+\frac{0.03}{N})^{Nt} f(0)$$

Now suppose the bank chops the time interval to infinitely small Δt and adds interest every Δt instance, i.e. compounded continuously. Or equivalently the bank pays interest infinitely many times in a year, so N is infinitely large in the equation above, we claim that

$$(1+\frac{0.03}{N})^{N} \approx e^{0.03}$$

Let us show why this is true.

Let's prove for the general case

$$(1+\frac{x}{N})^{N} \approx e^{x}$$

for N infinitely large and x is any number, not necessary 0.03.

Recall the definition of e^x is that $e^0 = 1$ and the instantaneous rate of e^x is e^x, namely

$$\frac{e^{x+\Delta x} - e^{x}}{\Delta x} = e^{x}$$

for Δx infinitely small. Now let's show $(1+\frac{x}{N})^N$ has these two properties,

$$(1+\frac{0}{N})^{N} = 1^{N} = 1$$

and we need to compute the rate

$$\frac{(1+\frac{x+\Delta x}{N})^N-(1+\frac{x}{N})^N}{\Delta x}$$

Using $a^N - b^N = (a-b)[\underbrace{a^{N-1}+a^{N-2}b+...+ab^{N-2}+b^{N-1}}_{N\ terms}]$, the rate is

$$\frac{1}{\Delta x}(\frac{\Delta x}{N})[(1+\frac{x+\Delta x}{N})^{N-1}+(1+\frac{x+\Delta x}{N})^{N-2}(1+\frac{x}{N})...+(1+\frac{x}{N})^{N-1}]$$

Since $\Delta x \ll N$ (N is infinitely large), we pretend that $\Delta x/N \approx 0$. The N terms in the sum are the same, so the rate becomes

$$\frac{1}{\Delta x}(\frac{\Delta x}{N})N(1+\frac{x}{N})^{N-1} = (1+\frac{x}{N})^{N-1}$$

Now we consider

$$\frac{(1+\frac{x}{N})^N}{(1+\frac{x}{N})^{N-1}} = 1+\frac{x}{N}$$

Since $x \ll N$, we pretend that $x/N \approx 0$, hence

$$\frac{(1+\frac{x}{N})^N}{(1+\frac{x}{N})^{N-1}} \approx 1$$

So

$$(1+\frac{x}{N})^{N-1} \approx (1+\frac{x}{N})^N$$

Putting everything together, the rate of $(1+\frac{x}{N})^N$ is

$$\frac{(1+\frac{x+\Delta x}{N})^N-(1+\frac{x}{N})^N}{\Delta x} \approx (1+\frac{x}{N})^N$$

hence

$$e^x \approx (1+\frac{x}{N})^N$$

Therefore

$$f_{compounded\,cont}(t) = e^{0.03t}f(0)$$

For this problem we want

$$e^{0.03t} = 2$$

or

$$t = \frac{\ln 2}{0.03}$$

d.

$$\frac{f(3)-f(1)}{3-1} = \frac{1}{2}$$

Problem 4

a. From $\ln x$ and $\ln(x+3)$ terms, we need $x > 0$ and $x + 3 > 0$, so $x > 0$.

b.

$$f = \ln \frac{2x^2}{x+3}$$

with domain $x > 0$.

We set

$$\frac{2x^2}{x+3} = 1 \implies (2x-3)(x+1) = 0$$

So $x = 3/2$.

Problem 5

Simplify to

$$2\sin^2 x - \sin x - 1 = (\sin x - 1)(2\sin x + 1) = 0$$

So

$$\sin x = 1 \implies x = \pi/2, \pi/2 + 2\pi = 5\pi/2$$

The other term is hard. You should draw $\sin x$ and see that there are 4 roots. First solve it in the usual branch $[0, \pi]$.

$$\sin x = -1/2 \implies \sin(-x) = 1/2 \implies -x = \pi/6, \text{ and } -x = \pi - \pi/6 = 5\pi/6$$

so

$$x = -\pi/6 + 2\pi = 11\pi/6$$

and

$$x = 11\pi/6 + 2\pi = 23\pi/6$$

and the other two

$$x = -5\pi/6 + 2\pi = 7\pi/6$$

and

$$x = 7\pi/6 + 2\pi = 19\pi/6$$

Hence total 6 roots

$$\pi/2,\ 5\pi/2,\ 7\pi/6,\ 11\pi/6,\ 19\pi/6,\ 23\pi/6$$

Problem 6

a. Set $y = 0$, so $2 - x = 0$ or $x = 2$

b. Set $x = 0$, so $y = e^2 - 1$

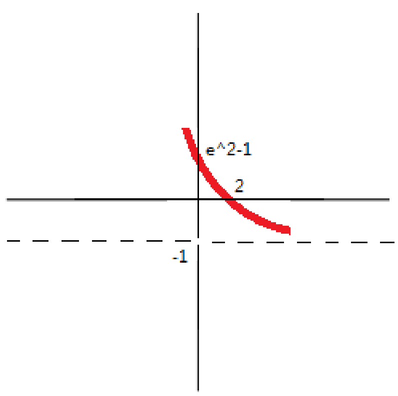
c. Shift e^x to the left by 2 units, we get e^{x+2}, then flip the graph about y axis, get e^{-x+2}, then move the graph down by 1 unit, get $e^{-x+2} - 1$. [note: one can also do the flipping first , i.e. $f(x) \to f(-x)$, so e^x become e^{-x}, then there is the cache. Then shift the graph to the right, not to the left, by 2 units, because reflection and translation don't commute in general. We get $e^{-(x-2)} = e^{2-x}$.]

d. [cf problem 2(c) above] $f \to -1$

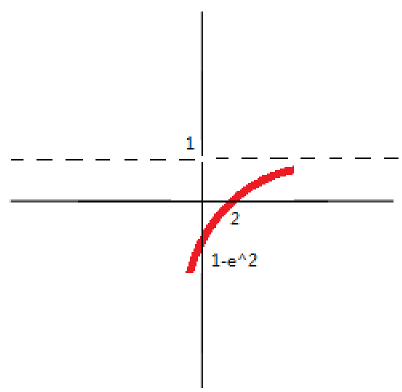
e. $f \to \infty$

f. Use (c)

g. flip (f) about the y axis



(f)



(g)

Problem 7

$$f = 2\sin(2x) - 1$$

a. amplitude $= 2$, which is half of the distance of oscillation. Period $= \pi$, because

$$f(x+\pi) = 2\sin(2x+2\pi) - 1 = 2\sin(2x) - 1 = f(x)$$

b. We want $\sin(2x) = 1/2 \implies 2x = \pi/6$. Using the period of the function, we can pick two values to be

$$\pi/12,\ \pi/12 + \pi = 13\pi/12$$

c. f is minimum, when $\sin(2x)$ is minimum, hence

$$\sin 2x = -1 \implies 2x = -\pi/2$$

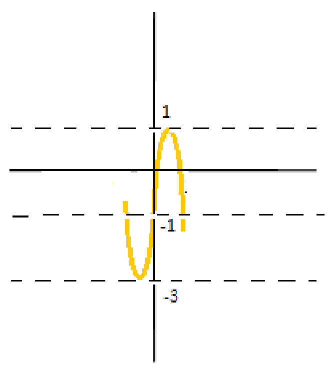
so we can pick two values to be

$$-\pi/4,\ -\pi/4 + \pi = 3\pi/4$$

d. Rang

$$[-2-1, 2-1] = [-3, 1]$$

e. Draw $f = 2\sin 2x - 1$



Problem 8

a. First find out that $x_1 = -1$ is a root. So the other two roots by the long division

$$(x - x_2)(x - x_3) = \frac{x^3 - 7x - 6}{x + 1} = x^2 - x - 6$$

so $x_2 = -2$ and $x_3 = 3$.

b. Use

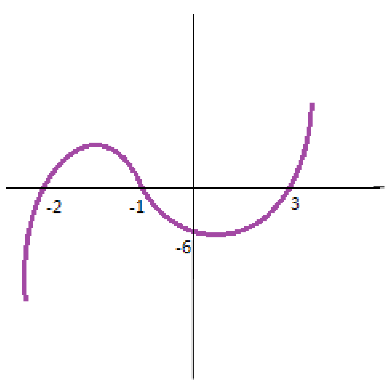
$$p(x) = (x + 2)(x + 1)(x - 3)$$

all three are simple roots, so the function goes positive/negative alternatively between the regions separated by the roots, e.g for $x > 3$, all three terms are positive, so the product $p(x) > 0$; for $-1 < x < 3$, two terms are positive and one term is negative, so $p(x) < 0$.

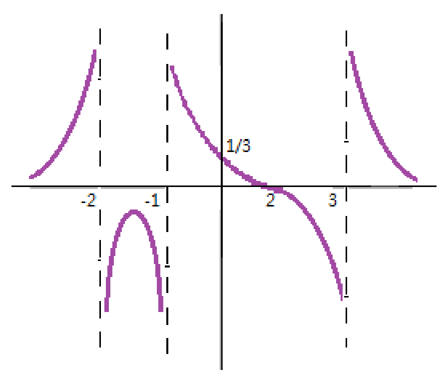
c. domain $D = \mathbb{R}/\{-2, -1, 3\}$

d. f blows up near -2, -1, and 3 so they are the vertical asymptotes. $f \to 0$ when $x \to \pm\infty$, so the horizontal asymptote is x axis.

e. f switches signs between regions $(-\infty, -2)$, $(-2, -1)$, $(-1, 2)$, $(2, 3)$, and $(3, \infty)$. And combining the asymptotic behaviors near the poles,



(b)



(e)

Problem 9

a. $\sin \frac{200+24+3}{4}\pi = \sin \frac{3}{4}\pi = \sin \frac{\pi}{4} = 1/2$, $\csc(227\pi/4) = 2$

b. Let $\tan x = 5$, now draw a right triangle with angle x, the opposite side has length 5, and the adjacent side has length 1, then $\sin x = 5/\sqrt{5^2+1} = 5/\sqrt{26}$

c. Draw a unit circle, and find a point (x, y) in the quadrant on the circle such that

$$\frac{y}{x} = 3$$

since $x^2 + y^2 = 1$, we can solve for x. We get

$$x = -1/\sqrt{10}$$

So

$$\sec t = \frac{1}{\cos t} = \frac{1}{x} = -\sqrt{10}$$

d. Let $\tan x = a$. And that $x \in (-\pi/2, \pi/2)$ is required for $\tan x$ to be invertible, i.e. we have

$$x = \arctan a$$

For $x \in (0, \pi/2)$, i.e. $a > 0$, we already know the answer from (b)

$$\sin x = \frac{a}{\sqrt{a^2+1}}$$

And when a becomes $-a$, x becomes $-x$, so $\sin x$ becomes $-\sin x$, hence the above expression works for $a < 0$ as well.