ON THE BETTI NUMBERS OF NODAL SETS OF THE ELLIPTIC EQUATIONS

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Abstract. In this paper we study the topological properties of the nodal sets, $\mathcal{N}(u) = \{x : u(x) = 0\}$, of solutions, u, of second order elliptic equations. We show that the total Betti number of a nodal set can be controlled by the coefficients and certain numbers of their derivatives of equations as well as the vanishing order of the corresponding solution.

1. Introduction

The main object of this paper is to study the topological complexity of the nodal sets of solutions to general elliptic equations. We will in particular study the Betti numbers of nodal (or general level sets when the zeroth order of equations vanish) sets.

The geometric structure and measure of the nodal sets of real-valued solutions to the linear elliptic equations have been studied by many authors, see the work of Oleinik-Petrovskii [24], Cheng ([5]), Donnelly-Fefferman ([7], [8]), Garofalo-Lin ([11]), Han ([13]), Han-Hardt-Lin ([14]), Hardt-Simon ([15]), Lin([17], [18]) and more recent ones, see for examples, Cheeger et al ([4]), Colding-Minicozzi ([6]), Hamid-Sogge ([12]), Sogge-Zelditch ([26]). There are also some results for the nodal sets of complex-valued or vector-valued solutions, see for instance the work of Berger-Rubinstein ([3]), Elliott-Matano-Tang ([9]), C. Bär ([1]), Helffer-Hoffmann-Hoffmann-Owen ([16]) and X.B. Pan ([23]).

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The above works were focused mainly on the following issues: the Hausdorff dimension and the Hausdorff measure estimates and the local smooth structure of nodal and critical point sets. For the topological complexity of the nodal sets, say the Betti numbers, there were only a few results known to authors, Y.Yomdin ([28], [29],[30]). There were, however, well-known works concerning polynomials done by R. Thom [27] and J. Milnor [21]. In [21], an upper bound for the sum of the Betti numbers of a real algebraic variety was proved by using the Morse theory.

Lemma 1.1. ([21]) *Let $f(x)$ be a degree N polynomial of $x \in \mathbb{R}^n$, then*

$$\textit{the total Betti number of } f^{-1}(0) \leq c(n)N^n.$$

This result shows that the total Betti number can be bounded by a constant depending on the degree of the polynomial and the number of unknowns. From earlier analysis for nodal and critical point sets of solutions of elliptic equations, one tends to believe that the analogy may be also valid for bounds on Betti numbers. Indeed, solutions of elliptic equations possess properties similar to that of analytic functions. Naturally, the first step one needs to do would be to control the so-called "generalized degrees" of solutions. This generalized degree has a natural substitute, the Almgren's frequency, which has already been applied in the study of the quantitative unique continuation property of solutions, see for example, [11]. The next important step would be to get a more quantified version of Morse Theory that J.Milnor had used in the above lemma. Very much like a quantitative version of Stability Lemma in Han-Hardt-Lin [14] which played a critical role in obtaining the geometric measure estimates for critical sets of solutions, the compactness of solutions of second order elliptic equations with an uniformly bounded vanishing order would be crucial. In order for a suitable perturbation argument to work which is necessary for understanding solutions of elliptic equations which are in general non-analytic, one also requires coefficients of equations to be uniformly elliptic and to be of certain fixed number of orders of smoothness.

Let us first introduce some notations. Throughout this paper, we consider the general elliptic equation,

$$\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b_i(x)D_i u(x) + c(x)u(x) = 0 \quad \text{in } B_1(0) \subset \mathbb{R}^n, \tag{1.1}$$

where the coefficients satisfy the following assumptions:

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \ \forall \xi \in \mathbb{R}^3, \ x \in B_1(0), \tag{1.2}$$

and for a suitably large M

$$\sum_{i,j=1}^{n} \|a_{ij}\|_{C^{M,\alpha}(B_1(0))} + \sum_{i=1}^{n} \|b_i\|_{C^{M,\alpha}(B_1(0))} + \|c\|_{C^{M,\alpha}(B_1(0))} \leq K. \tag{1.3}$$

where λ, Λ and K are positive constants.

The following theorem is the main result of the paper:

Theorem 1.2. *Let u be the solution of problem (1.1), and the coefficients a_{ij}, b_i and c satisfy the conditions (1.2) and (1.3). Here M may chosen to be $2(2N_0)^n$. Then for $B_{\frac{1}{2}}(0)$,*

$$\text{the total Betti numbers of } \mathcal{N}(u) \cap B_{\frac{1}{2}}(0) \leq C(n, K, \lambda, \Lambda, N_0).$$

where the constant N_0 is the upper bound of the vanishing order of u at any point of $B_{\frac{1}{2}}(0)$.

In Section 3 below, we will give the precise definition of the vanishing order and the existence of an uniformly upper bound for N_0.

This paper is organized as follows. In Section 2, we prove some stability results, which play a crucial role in the proof of Theorem 1.2. In Section 3, after recalling the definition of the vanishing order of u, we give a natural definition of the Betti numbers of $\mathcal{N}(u) \cap B_{\frac{1}{2}}(0)$. As the nodal sets may not be in general a smooth hypersurface, the notation of its Betti numbers have to be defined properly. We end with this section the proof of the main theorem.

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2. STABILITY RESULTS

In this section we prove some stability results, which will be used in the proof of Theorem 1.2. First, we prove a general statement 2.1 which is somewhat standard in the study of stability of singularities in the algebra geometry. Given some smooth map g, the stability result holds for any smooth map sufficiently close to g with the closeness measured by several constants all depending on g. If we take the map g as a polynomial map in a suitable family that comes naturally from the Taylor expansions of solutions to a wide class of elliptic equations with an uniform bound on the vanishing orders, we get a stronger stability statement in Lemma 2.3. Roughly speaking, given N_0, for all (L-harmonic) polynomial maps g with degrees less than $2N_0$ and with the Almgren frequencies of g bounded by N_0, then the constants can be chosen to depend only on N_0 and the dimension such that the conclusion of the stability lemma holds.

We start with the following lemma:

Lemma 2.1. *Let $G(x) = (g_1(x), \cdots, g_n(x))$ be a smooth map defined from $B_1(0) \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Assume that the extended map $G(x)$ from the unit ball in $\mathbb{C}^n$ to $\mathbb{C}^n$ is holomorphic. If*

$$\mathcal{H}^0\{G^{-1}(0) \cap B_1(0)\} = m < \infty,$$

then there exist positive constants δ_*, M_*, r_* and C_* all depending on the function $G(x)$, such that for any $F(x) = (f_1(x), \cdots, f_n(x)) \in C^{M_*}(B_1(0))$ with

$$\|F - G\|_{C^{M_*}(B_1(0))} < \delta_*,$$

we have

$$\mathcal{H}^0\left\{F^{-1}(0) \cap B_{r_*}(0)\right\} \leq C_*$$

with multiplicity.

Proof. The proof is given in [14, Theorem 4.1]. For the sake of the completeness, we sketch it here. Assume that the nodal set of $G = (g_1, g_2, \cdots, g_n)$ contains m points, letting

$$\mathcal{N}(G) = \mathcal{H}^0\left\{G^{-1}(0) \cap B_1(0)\right\} = \{0, x_1, \cdots, x_{m-1}\}.$$

Step 1. For $0 \in \mathcal{N}(G)$, one can show that there is an integer N_j^0 and a holomorphic function $a_{ij}^0(x)$ such that,

$$x_j^{N_j^0} = \sum_{i=1}^n a_{ij}^0(x) g_i(x), \text{ for each } j = 1, 2, \cdots, n,$$

where $x \in \mathbb{R}^n$.

Letting $\Lambda_0 = n \max\{N_1^0, N_2^0, \cdots, N_n^0\}$, then any monomial in $x_1, \cdots, x_n$ of degree $\geq \Lambda_0$ belongs to the ideal (G) near 0. Therefore, we can choose an integer μ_0 such that

$$\dim_{\mathbb{R}} P^{\mu_0}/(G) \leq \mu_0,$$

where the set $P^{\mu_0}(\mathbb{R}^n)$ denotes the collection of all polymomials in $\mathbb{R}$ of degree μ_0.

Following the arguments in the proof of Theorem 4.1 ([14]), we obtain that there exist neighborhoods U_0, V_0 and Q_0 of the origin with $V_0 \subset U_0$ and $G(V_0) \subset Q_0$ such that for any $a \in C^{2\mu_0}(U_0)$ there exist $\alpha_1^0, \alpha_2^0, \cdots, \alpha_n^0 \in C^1(Q_0)$ such that

$$a(x) = \sum_{i=1}^{\mu_0} e_i(x) \alpha_i^0(g(x)), \text{ for } x \in V_0. \tag{2.1}$$

Moreover, there exists a $\delta_0 > 0$, such that for any $F \in C^{2\mu_0}(U_0)$ with $\|F - G\|_{C^{2\mu_0}(U_0)} < \delta_0$, the identity (2.1) still holds with G replaced by F.

As in ([14]), there exists $r_0 > 0$, such that F has μ_0 zeros at most in $B_{r_0}(0)$.

Step 2. For each $x_k \in \mathcal{N}(G)$, we repeat the step 1 to get corresponding μ_k, U_k, V_k, Q_k, δ_k and r_k, such that for F with $\|F - G\|_{C^{2\mu_k}(U_k)} < \delta_k$, F has μ_k zeros at most in $B_{r_k}(x_k)$.

Therefore, by taking

$$\delta_* = \min\{\delta_0, \cdots, \delta_{m-1}\}, \quad M_* = 2 \sum_{k=0}^{m-1} \mu_k, \quad C_* = \sum_{k=0}^{m-1} \mu_k,$$

and choosing r_* such that

$$B_{r_*}(0) \subset \bigcup_{k=0}^{m-1} B_{r_k}(x_k) \subset \bigcup_{k=0}^{m-1} V_k,$$

we have,

$$\mathcal{H}^0 \left\{ F^{-1}(0) \cap B_{r_*}(0) \right\} \leq C_*.$$

for $F(x) \in C^{M_*}(B_1(0))$ with $\|F-G\|_{C^{M_*}(B_1(0))} < \delta_*$. To end our proof, we remark that the constants M_*, δ_*, r_* and C_* all depend on $G(x)$. $\square$

Remark 1. In general, the constant M_* may be very large, which means the function $F(x)$ must have sufficiently higher orders of derivatives. It is the reason why we need the assumption (1.3) to make sure solutions of such equations would be also sufficiently smooth for the conclusion of Theorem 1.2 to be valid. On the other hand, if G is a polynomial of degree at most N, then from the above proof of ([14]), M_* may be chosen to be not bigger than $2N^n$.

Remark 2. *Let us introduce the set*

$$\mathcal{G}_{N_0} = \{g(x) : B_1(0) \to \mathbb{R}^1, \textit{a polynomial in } B_1(0) \subset \mathbb{R}^n, \textit{ with degree} \le 2N_0, \ \|g\|_{L^2} = 1,$$
$$\int_{B_1(0)} |\nabla g|^2 dx \le N_0\}.$$

Then $\mathcal{G}_{N_0}$ is a compact set in the polynomial space of C^{M_*}, see for example, [14].

For given numbers ε and θ in $[0,1]$ we also define the set

$$\mathcal{F}_{N_0} = \{f : \ f = g^2 + \varepsilon^2|x|^2 - \theta^2, \ g \in \mathcal{G}_{N_0}\}.$$

Then $\mathcal{F}_{N_0}$ is also compact.

Corollary 2.2. *For a given $f_0 \in \mathcal{F}_{N_0}$, there exist positive constants M_0, r_0, δ_0 and C_0, such that if $f \in C^{M_0+1}(B_1(0))$ with $\|f - f_0\|_{C^{M_0+1}} < \delta_0$, and the hypersurfaces $\{x : \ f(x) = 0\}$ and $\{x : f_0(x) = 0\}$ are both regular, then*

$$\mathcal{H}^0 \left\{F^{-1}(0) \cap B_{r_0}(0)\right\} \le C_0,$$

where F is a vector feild defined by $F(x) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n-1}}, f)$, for a suitable choice of rectangular coordinate system $(x_1, x_2, \cdots, x_n)$ of $\mathbb{R}^n$.

Proof. First, Let us use the following notations for the two hypersurfaces:

$$W_1 = \{x \in B_1(0) : f_0(x) = 0\}, \quad W_2 = \{x \in B_1(0) : f(x) = 0\}.$$

We use $\vec{n}_1$ and $\vec{n}_2$ to denote the Gaussian maps of W_1 and W_2. Sard's theorem ensures that the sets

$$\{\vec{n}_1 : \nabla \vec{n}_1 = 0\} \text{ and } \{\vec{n}_2 : \nabla \vec{n}_2 = 0\}$$

both have measure zero in $\mathbb{S}^{n-1}$.

Up to some coordinate rotation, we may choose directions $(0, \cdots, 0, 1) \in \mathbb{R}^n$, such that $\vec{n}_1(q) = (0, \cdots, 0, 1)$, and $(0, \cdots, 0, 1)$ are neither the critical values of $\vec{n}_1$ nor the critical values of $\vec{n}_2$.

Therefore, the height function $(x_1, \cdots, x_n) \to x_n$ of the hypersurfaces W_1 and W_2 has no degenerate critical points. And the critical points of the height function on W_1 and W_2 can be characterized as the solutions of the equations

$$F_0(x) = (\frac{\partial f_0}{\partial x_1}, \cdots, \frac{\partial f_0}{\partial x_{n-1}}, f_0) = 0, \quad F(x) = 0.$$

Since f_0 is a polynomail of degree $\leq 2N_0$, we can apply the result of [21] to obtain that

$$\mathcal{H}^0 \left\{ F_0^{-1}(0) \cap B_1(0) \right\} = m_0 < \infty.$$

For any $f \in C^{M_0+1}(B_1(0))$, we have $\|F - F_0\|_{C^{M_0}(B_1(0))} < \delta_0$. Applying Lemma 2.1 to the maps F and F_0, there exist positive constants r_0 and C_0 depending on f_0, such that

$$\mathcal{H}^0 \left\{ F^{-1}(0) \cap B_{r_0}(0) \right\} \leq C_0.$$

□

The next Stability Lemma is the key to our proof of the main theorem.

Lemma 2.3. *There exist positive constants δ_0, M_0, r_0 and C_0 only depending on N_0, such that if $f \in C^{M_0+1}(B_1(0))$, and if the hypersurface $\{x : f(x) = 0\}$ is regular with $\|f - g\|_{C^{M_0+1}(B_1(0))} < \delta_0$ and $g \in \mathcal{F}_{N_0}$, satisfying that $\{x : g = 0\}$ is also a regular hypersurface, then we have*

$$\mathcal{H}^0 \left\{F^{-1}(0) \cap B_{r_0}(0)\right\} \leq C_0,$$

where the map $F(x) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n-1}}, f)$, for a suitable choice of rectangular coordinate system $(x_1, x_2, \cdots, x_n)$ of $\mathbb{R}^n$.

Proof. From Corollary 2.2, we can see that for each $g \in \mathcal{F}_{N_0}$, there exist positive constants δ_g, M_g, r_g and C_g, such that if $\|g - f\|_{C^{M_g+1}(B_1(0))} < \delta_g$, then

$$\mathcal{H}^0 \left\{F^{-1}(0) \cap B_{r_g}(0)\right\} \leq C_g.$$

Let us remark that the constants M_g and C_g for any g can be chosen uniformly depending only on N_0, say $2(2N_0)^n$.

Therefore, $\{\mathcal{O}(g, \frac{\delta_g}{2}) : g \in \mathcal{F}_{N_0}\}$, form an open cover of $\mathcal{F}_{N_0}$. By the compactness of $\mathcal{F}_{N_0}$, there exists a finite cover, denoted by

$$\mathcal{F}_{N_0} \subset \bigcup_{j=1}^{k} \mathcal{O}(g_j, \frac{\delta_0}{2}).$$

Let us denote the representatives by $\mathcal{R}_{\mathcal{F}} = \{g_1, \cdots, g_k\}$, and we take

$$\delta_0 = \frac{1}{2} \min\{\delta_1, \cdots, \delta_k\}, \quad r_0 = \min\{r_1, \cdots, r_k\}.$$

We claim that the constants δ_0 and r_0 only depend on N_0. In other words, for any $g \in \mathcal{F}_{N_0}$, the stability results holds for δ_0 and r_0.

In fact, for any given $g \in \mathcal{F}_{N_0}$, by the above covering of $\mathcal{F}_{N_0}$, there exists some $g_s \in \mathcal{R}_{\mathcal{F}}$, such that $\|g_s - g\|_{C^{M_0+1}} < \frac{\delta_s}{2}$. If $f \in C^{M_0+1}(B_1(0))$ with $\|f - g\|_{C^{M_0+1}(B_1(0))} < \delta_0$, then

$$\|f - g_s\|_{C^{M_0+1}(B_1(0))} < \delta_0 + \frac{\delta_s}{2} \leq \frac{\delta_s}{2} + \frac{\delta_s}{2} = \delta_s.$$

Applying Corollary 2.2 to g_s, we get

$$\mathcal{H}^0\left\{F^{-1}(0) \cap B_{r_0}(0)\right\} \subset \mathcal{H}^0\left\{F^{-1}(0) \cap B_{r_s}(0)\right\} \leq C_0.$$

Hence the stability result works uniformly for the fixed constants δ_0 and r_0, which only depend on N_0.

Therefore, if $f \in C^{M_0+1}(B_1(0))$ with $\|f - g\|_{C^{M_0+1}(B_1(0))} < \delta_0$ for some $g \in \mathcal{F}_{N_0}$, we may conclude that

$$\mathcal{H}^0\left\{(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n-1}}, f)^{-1}(0) \cap B_{r_0}\right\} \leq C_0,$$

which completes the proof. □

3. Estimates on the total Betti numbers

We can now give an upper estimate of the total Betti numbers of a nodal set. To begin with, let us recall the definition of vanishing order of u.

Assume that $u \in W^{1,2}(B_1(0))$ is a solution of (1.1) in $B_1(0) \subset \mathbb{R}^n$. To define the vanishing order of the weak solution u, we only need the following weaker assumptions on the coefficients than (1.2) and (1.3):

$$\begin{aligned}
&\sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \ \forall \xi \in \mathbb{R}^3, \ x \in B_1(0), \\
&\sum_{i,j=1}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_i| + |c| \leq \kappa, \ \forall \ x \in B_1(0), \qquad (3.1)
\end{aligned}$$

$$\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| \le L|x-y|, \ \forall\ x, y \in B_1(0),$$

for some positive constants λ, κ and L.

In [11], the authors define,

$$H(p,r) = \int_{\partial B_r(p)} \mu u^2 \mathrm{d}x,$$

$$D(p,r) = \int_{B_r(p)} a^{ij}(x) u_{x_i} u_{x_j} \mathrm{d}x,$$

where the positive function μ is constructed in [11].

Letting

$$N(p,r) = \frac{rD(p,r)}{H(p,r)},$$

a monotonicity property for $N(p,r)$ with respect to r is proved in [11]: there exist positive constants $r_0 = r_0(n, \lambda, \kappa, L)$ and $\theta = \theta(n, \lambda, \kappa, L)$ such that the function $N(p,r)\exp(\theta r)$ is monotone nondecreasing in $(0, r_0)$.

Therefore, one can define at each point $p \in B_{\frac{1}{2}(0)}$ the quantity,

$$\mathcal{O}_u(p) := \lim_{r \to 0} N(p,r).$$

We shall call it the vanishing order of u at point p. The following lemma says that the vanishing order of u in $B_{\frac{1}{2}}(0)$ is uniformly bounded. For details see [17]. In our paper, we denote the uniformly upper bound by the constant N_0.

Lemma 3.1 ([17]). *Set*

$$N = \frac{\int_{B_1(0)} |Du|^2 \mathrm{d}x}{\int_{\partial B_1(0)} u^2 \mathrm{d}x}.$$

Then the vanishing order of u at any point of $B_{1/2}(0)$ is never exceeded by $C(n, \lambda, \kappa, L)N$.

The following lemma comes from [13] and [14]. It yields a decomposition result and Schauder estimates of the error term.

Lemma 3.2 ([13], [14]). *Suppose that u is the solution to the equation (1.1). Under the assumption (3.1), for any $y \in B_{1/2}(0)$, the solution u has the following decomposition:*

$$u(x) = P_d(x-y) + \phi(x), \quad x \in B_{1/4}(y), \tag{3.2}$$

where the nonzero polynomial $P_d(x-y)$ is homogeneous of degree $d = d(y) \le N_0$ satisfying

$$\sum_{i,j=1}^{n} a_{ij}(y) D_{ij} P_d = 0, \tag{3.3}$$

and

$$|P_d(x-y)| \le C|x-y|^d, \quad x \in B_{1/4}(y). \tag{3.4}$$

Moreover ϕ satisfies the following estimate: for some $\alpha \in (0,1)$,

$$|\phi(x)| \le C|x-y|^{d+\alpha}, \quad x \in B_{1/4}(y). \tag{3.5}$$

The constant C depends on $n, N_0, \lambda, \kappa, L$.

Furthermore, if we assume that the assumptions (1.2) and (1.3) hold, by using the interior Schauder estimates one can get,

$$\begin{aligned} &|D^i\phi(x)| \le \tilde{C}|x-y|^{d-i+\alpha}, \quad \text{for } i = 1, \cdots, d,\\ &|D^i\phi(x)| \le \tilde{C}, \qquad \text{for } i = d+1, \cdots, \tilde{M}+1, \end{aligned} \tag{3.6}$$

for all $x \in B_{1/8}(y)$. The positive constant $\tilde{C}$ depends on $n, N_0, \lambda, \Lambda, K$.

Next, let us use the definition of the total Betti number of $\mathcal{N}(u) \cap B_{1/2}(0)$ in [21]:

Definition 3.1. *The total Betti numbers of $\mathcal{N}(u) \cap B_{1/2}(0)$ is defined to be*

$$\textit{the } \limsup_{(\varepsilon,\theta)\to(0^+,0^+)} \textit{ of the total Betti numbers of } \mathcal{N}_{(\varepsilon,\theta)}(u)\cap B_{1/2}(0),$$

where $\mathcal{N}_{(\varepsilon,\theta)}(u)\cap B_{1/2}(0)=\{x\in\mathbb{R}^n:\ u^2+\varepsilon^2|x|^2=\theta^2\}$.

Before we prove Theorem 1.2, let us also recall the weak Morse inequalities (see [22]).

Lemma 3.3 ([22])**. (Weak Morse Inequalities)** *Let b_k be the k-th Betti number of a compact manifold M and c_k denote the set of the critical points of index k of a Morse function on M, then*

$$b_k \le \#c_k, \quad \textit{for all } k\in[0,\mathrm{dim}M],$$

and

$$\sum_{k=0}^{\mathrm{dim}M}(-1)^k\#c_k=\sum_{k=0}^{\mathrm{dim}M}(-1)^k b_k=\chi(M).$$

By the above lemma, it is therefore sufficient for us to do the following two things. One is to construct suitable Morse functions. The other is to get a suitable estimate of the numbers of critical points of such Morse functions.

The following lemma comes from [21]. It gives an estimate on the number of zeros of polynomial equations.

Lemma 3.4 ([21])**.** *Let $V_0\subset\mathbb{R}^m$ be a zero-dimensional variety defined by polynomial equations $f_1=0,\cdots,f_m=0$. Suppose that the gradient vectors $df_1,\cdots,df_m$ are linearly independent at each point of V_0. Then the number of points in V_0 is at most equal to the product $(\mathrm{deg}f_1)(\mathrm{deg}f_2)\cdots(\mathrm{deg}f_m)$.*

Now we can proceed with **the proof of Theorem 1.2:** Taking any fixed point $y_0\in\mathcal{N}(u)\cap B_{1/2}(0)$, there exists a nonzero homogeneous polynomial P_d of degree $0\le d\le N_0$

such that

$$u(x) = P_d(x - y_0) + \phi(x), \quad x \in B_{1/4}(y_0), \tag{3.7}$$

where P_d and ϕ satisfy the properties (3.3)-(3.6).

From (3.7), we have the decomposition of u^2,

$$u^2(x) = P_{2d}(x - y_0) + \tilde{\phi}(x), \tag{3.8}$$

where $\tilde{\phi}(x) = 2P_d(x - y_0)\phi(x) + \phi^2(x)$ and $P_{2d}(x - y_0) = P_d^2(x - y_0)$. Then $P_{2d}(x - y_0)$ is a nonzero homogenous polynomial of degree $2d$. Note however, the equation (3.3) may not be satisfied by u^2, but it does not effect the compactness of set formed by those $P_{2N_0}(x - y_0)$'s.

Given two positive numbers ε and θ, let us introduce some notations first:

$$K_{y_0}(\varepsilon, \theta) = \left\{x \in \mathbb{R}^n : u^2 + \varepsilon^2|x - y_0|^2 \leq \theta^2\right\},$$

$$W_{y_0}(\varepsilon, \theta) = \left\{x \in \mathbb{R}^n : P_{2d}(x - y_0) + \varepsilon^2|x - y_0|^2 \leq \theta^2\right\},$$

$$\partial K_{y_0}(\varepsilon, \theta) = \left\{x \in \mathbb{R}^n : u^2 + \varepsilon^2|x - y_0|^2 = \theta^2\right\},$$

$$\partial W_{y_0}(\varepsilon, \theta) = \left\{x \in \mathbb{R}^n : P_{2d} + \varepsilon^2|x - y_0|^2 = \theta^2\right\}.$$

Since $K_{y_0}(\varepsilon, \theta)$ is contained in $\bar{B}_{\theta/\varepsilon}(y_0)$, $K_{y_0}(\varepsilon, \theta)$ is a compact set. The hypersurface $\partial K_{y_0}(\varepsilon, \theta)$ is nonsingular if and only if θ^2 is a regular value of the function $u^2 + \varepsilon^2|x - y_0|^2$. By Sard's theorem, for any fixed ε, almost all the values of θ are regular. Similarly, applying Sard's theorem to the function $P_{2d} + \varepsilon^2|x - y_0|^2$, we have that almost all the values of θ are regular as well. Henceforth, we can always assume that ε and θ are chosen so that both the hypersurface $\partial K_{y_0}(\varepsilon, \theta)$ and $\partial W_{y_0}(\varepsilon, \theta)$ are nonsingular. Moreover we may also assume that they are contained in $B_{1/2}(0)$.

Let $\vec{n}_W(x)$ and $\vec{n}_K(x)$ be the exterior unit normal vector fields of the hypersurfaces $\partial K_{y_0}(\varepsilon,\theta)$ and $\partial W_{y_0}(\varepsilon,\theta)$ respectively. Sard's theorem ensures that the sets

$$\{\vec{n}_W(x) : \nabla\vec{n}_W(x) = 0\} \text{ and } \{\vec{n}_K(x) : \nabla\vec{n}_K(x) = 0\}$$

both have measure zero in $\mathbb{S}^{n-1}$.

Therefore, up to some coordinate rotation, we can choose directions $(0,\cdots,0,1) \in \mathbb{R}^n$, such that $(0,\cdots,0,1)$ are neither the critical values of $\vec{n}_W(x)$ nor the critical values of $\vec{n}_K(x)$.

Let us denote the height function $(x_1,\cdots,x_n) \to x_n$ of the hypersurfaces $\partial K_{y_0}(\varepsilon,\theta)$ and $\partial W_{y_0}(\varepsilon,\theta)$ by h_W and h_K respectively. From Morse theory (for details see the proof of [21, Theorem 1]), we have that both h_W and h_K have no degenerate critical points.

Note that the critical point of h_W and h_K can be characterized as follows:

$$F(x-y_0) = \begin{pmatrix} \frac{\partial P_{2d}(x-y_0)}{\partial x_1} + \frac{\partial\tilde{\phi}(x)}{\partial x_1} + 2\varepsilon^2(x_1 - y_1^0) \\ \cdots \\ \frac{\partial P_{2d}(x-y_0)}{\partial x_{n-1}} + \frac{\partial\tilde{\phi}(x)}{\partial x_{n-1}} + 2\varepsilon^2(x_{n-1} - y_{n-1}^0) \\ P_{2d}(x-y_0) + \tilde{\phi}(x) + \varepsilon^2|x-y_0|^2 - \theta^2 \end{pmatrix} = 0,$$

$$G(x-y_0) = \begin{pmatrix} \frac{\partial P_{2d}(x-y_0)}{\partial x_1} + 2\varepsilon^2(x_1 - y_1^0) \\ \cdots \\ \frac{\partial P_{2d}(x-y_0)}{\partial x_{n-1}} + 2\varepsilon^2(x_{n-1} - y_{n-1}^0) \\ P_{2d}(x-y_0) + \varepsilon^2|x-y_0|^2 - \theta^2 \end{pmatrix} = 0.$$

Applying Lemma 3.4 to the polynomial $G(x-y_0)$, we obtain

$$\mathcal{H}^0\{G^{-1}(0) \cap B_R(y_0)\} \le C(d), \quad \text{for any } R \in (0,1). \tag{3.9}$$

We introduce the C^{M+1}-norm weighted with the radius R, denoted by $\|\cdot\|^*_{C^{M+1}(B_R(a))}$,

$$\|w\|^*_{C^{M+1}(B_R(a))} = \sum_{i=0}^{M+1} R^i \sup_{x\in B_R(a)} |D^i w(x)|.$$

Using the inequalities (3.5) and (3.6), we can choose small $R = R(y_0) < \frac{1}{8}$ such that

$$\begin{aligned}
&\left\| \frac{1}{R^{2d-1}} \left(F(x-y_0) - G(x-y_0)\right) \right\|^*_{C^{M_0+1}(B_R(y_0))} \\
\leq & \frac{1}{R^{2d-1}} \left(CR^{2d-1+\alpha} + RCR^{2d-2+\alpha} + \cdots + R^{2d-1}CR^{\alpha} + R^{2d}C + \cdots + R^{M_0+1}C \right) \\
\leq & CR^{\alpha} + CR + CR^2 + \cdots + CR^{M_0-2d+2} < \delta_0,
\end{aligned}$$

where δ_0 is the positive constant obtained in Lemma 2.3.

By the transformation $x \mapsto y_0 + Rx$, we compute

$$\begin{aligned}
&\left\| \frac{1}{R^{2d-1}} F(Rx) - \begin{pmatrix} \frac{\partial P_{2d}(x)}{\partial x_1} + 2\varepsilon^2 \frac{x_1}{R^{2d-2}} \\ \cdots \\ \frac{\partial P_{2d}(x)}{\partial x_{n-1}} + 2\varepsilon^2 \frac{x_{n-1}}{R^{2d-2}} \\ RP_{2d}(x) + \frac{\varepsilon^2|x|^2}{R^{2d-3}} - \frac{\theta^2}{R^{2d-1}} \end{pmatrix} \right\|_{C^{M_0+1}(B_1(0))} \\
=& \left\| \frac{1}{R^{2d-1}} \left[F(x-y_0) - G(x-y_0)\right] \right\|^*_{C^{M_0+1}(B_R(y_0))} \\
<& \delta_0.
\end{aligned}$$

Note that (3.9) is equivalent to

$$\mathcal{H}^0 \left\{ \begin{pmatrix} \frac{\partial P_{2d}(x)}{\partial x_1} + 2\varepsilon^2 \frac{x_1}{R^{2d-2}} \\ \cdots \\ \frac{\partial P_{2d}(x)}{\partial x_{n-1}} + 2\varepsilon^2 \frac{x_{n-1}}{R^{2d-2}} \\ RP_{2d}(x) + \frac{\varepsilon^2|x|^2}{R^{2d-3}} - \frac{\theta^2}{R^{2d-1}} \end{pmatrix}^{-1} (0) \cap B_1(0) \right\} \triangleq l(y_0) \leq C(N_0).$$

Therefore, we take the polynomial $g = RP_{2d}(x) + \frac{\varepsilon^2|x|^2}{R^{2d-3}} - \frac{\theta^2}{R^{2d-1}}$ in Lemma 2.3, then there exists a positive constant r_0 such that

$$\mathcal{H}^0\left\{\left[\frac{1}{R^{2d-1}}F(Rx)\right]^{-1}(0)\cap B_{r_0}(0)\right\}\leq C_0.$$

After transforming back to $B_R(y_0)$, one can get

$$\mathcal{H}^0\left\{\left[F(x-y_0)\right]^{-1}(0)\cap B_{r_0R}(y_0)\right\}\leq C_0.$$

Therefore, for any $r < r_0R$,

$$\mathcal{H}^0\left\{\left[F(x-y_0)\right]^{-1}(0)\cap B_{r}(y_0)\right\}\leq C_0.$$

Here we need to remark that the constant C_0 in the proof above also depends on n, K, λ, Λ, which come from the assumptions of the coefficients. For simplicity, we omit these constants in the notation C_0.

Recalling that the critical points of the height function h_K can be characterized by the zeros of $F(x - y_0)$. Therefore, from (3) we can see that the height function h_K, i.e. the Morse function of the hypersurface $\partial K_{y_0}(\varepsilon, \theta)$, only has finite critical points.

By Morse theory, the qth Betti number of $\partial K_{y_0}(\varepsilon, \theta)$ means the rank of the Čech cohomology group $H^q(\partial K_{y_0}(\varepsilon, \theta))$. Let

$$H^*(\partial K_{y_0}(\varepsilon,\theta)) = \bigoplus_q H^q(\partial K_{y_0}(\varepsilon,\theta)),$$

then the total Betti numbers of $\partial K_{y_0}(\varepsilon, \theta)$ is equals to rank $H^*(\partial K_{y_0}(\varepsilon, \theta))$.

The weak Morse inequalities imply

$$\begin{aligned}\text{rank } H^*(\partial K_{y_0}(\varepsilon,\theta)) \leq& \#\{\text{all the critical points of } h_K\}\\ \leq& \mathcal{H}^0\left\{\left[F(x-y_0)\right]^{-1}(0)\cap B_{r}(y_0)\right\}\end{aligned}$$

$$\leq C_0.$$

On the other hand, the Alexander duality theorem gives

$$\text{rank } H^*(K_{y_0}(\varepsilon,\theta)) = \frac{1}{2}\text{rank } H^*(\partial K_{y_0}(\varepsilon,\theta)).$$

Hence,

$$\text{rank } H^*(K_{y_0}(\varepsilon,\theta)) \leq C_0.$$

Now we can choose suitable sequences $\{\varepsilon_i\}$ and $\{\theta_i\}$, such that $\{\varepsilon_i\}$ decreases monotonely with the limit zero and $\{\theta_i/\varepsilon_i\}$ decreases monotonely with limit $r(y_0)$. Meanwhile each $\partial K_{y_0}(\varepsilon_i,\theta_i)$ is a nonsingular hypersurface.

It is easy to see that

$$K(\varepsilon_1,\theta_1) \supset K(\varepsilon_2,\theta_2) \supset \cdots \supset K(\varepsilon_n,\theta_n) \supset \cdots ,$$

and

$$\mathcal{N}(u) \cap B_{r(y_0)}(y_0) = \bigcap_{i=1}^{+\infty} K(\varepsilon_i,\theta_i).$$

Therefore,

$$\text{rank } H^*(\mathcal{N}(u) \cap B_{r(y_0)}(y_0)) \leq \limsup \text{rank } H^*(K_{y_0}(\varepsilon_i,\theta_i)) \leq C_0.$$

It is obvious that the collection of $\left\{B_{r(y)}(y) : y \in \mathcal{N}(u) \cap B_{1/2}(0)\right\}$ covers $\mathcal{N}(u) \cap B_{1/2}(0)$. By the compactness of $\mathcal{N}(u) \cap B_{1/2}(0)$, there exist finite balls $\{B_{r_j}(y_j) : j = 1, \cdots, \nu\}$ such that

$$\mathcal{N}(u) \cap B_{1/2}(0) \subset \bigcup_{j=1}^{\nu} B_{r_j}(y_j),$$

and for each j,

$$\text{rank } H^*(\mathcal{N}(u) \cap B_{r_j}(y_j)) \leq C_0.$$

Therefore,

$$\text{the total Betti numbers of } \mathcal{N}(u) \cap B_{1/2}(0) \leq C(N_0, K, \lambda, \Lambda, n),$$

which completes the proof of the main theorem. $\square$

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