Global Small Solutions to an MHD-Type System: The Three-Dimensional Case

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Abstract

In this paper, we consider the global well-posedness of a three-dimensional incompressible MHD type system with smooth initial data that is close to some nontrivial steady state. It is a coupled system between the Navier-Stokes equations and a free transport equation with a universal nonlinear coupling structure. The main difficulty of the proof lies in exploring the dissipative mechanism of the system due to the fact that there is a free transport equation of ϕ in the coupled equations and only the horizontal derivatives of ϕ is dissipative with respect to time. To achieve this, we first employ anisotropic Littlewood-Paley analysis to establish the key $L^1(\mathbb{R}^+;\mathrm{Lip}(\mathbb{R}^3))$ estimate to the third component of the velocity field. Then we prove the global well-posedness to this system by the energy method, which depends crucially on the divergence-free condition of the velocity field. © 2013 Wiley Periodicals, Inc.

1 Introduction

In this paper, we investigate the global well-posedness of the following three-dimensional incompressible system, which will be called of MHD type:

$$
(1.1)\qquad
\begin{cases}
\partial_t\phi + u\cdot\nabla\phi = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\partial_t u + u\cdot\nabla u - \Delta u + \nabla p & \\
\quad = -\operatorname{div}[\nabla\phi\otimes\nabla\phi], & \\
\operatorname{div} u = 0, & \\
\phi|_{t=0} = \phi_0,\ u|_{t=0} = u_0, &
\end{cases}
$$

with initial data (ϕ_0, u_0) smooth and close enough to the equilibrium state $(x_3, 0)$.

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Recall that the MHD system in $\mathbb{R}^d$ reads

$$
\begin{cases}
\partial_t b + u\cdot\nabla b = b\cdot\nabla u, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\
\partial_t u + u\cdot\nabla u - \Delta u + \nabla p\\
\quad = -\frac{1}{2}\nabla|b|^2 + b\cdot\nabla b,\\
\operatorname{div} u = \operatorname{div} b = 0,\\
b|_{t=0} = b_0,\ u|_{t=0} = u_0,
\end{cases}
\tag{1.2}
$$

where u, b denotes the flow velocity field and the magnetic field vector, respectively, and p the scalar pressure. This MHD system (1.2) with zero diffusivity in the equation for the magnetic field can be applied to model plasmas when the plasmas are strongly collisional or when the resistivity due to these collisions is extremely small. It often applies to the case when one is interested in the k-length scales that are much longer than the ion skin depth and the Larmor radius perpendicular to the field, long enough along the field to ignore the Landau damping, and time scales much longer than the ion gyration time [3, 7, 11]. In the particular case when $d = 2$ in (1.2), $\operatorname{div} b = 0$ implies the existence of a scalar function ϕ so that $b = (-\partial_2\phi\ \ \partial_1\phi)^{\mathsf{T}}$, and the corresponding system becomes (1.1) with $d = 2$.

The goal of this paper is to solve the global small solutions to the three-dimensional case of (1.1). We believe the techniques developed in this paper will be useful for various important and related problems. In fact, by combining the ideas and techniques developed in this paper, some additional estimates and a new formulation of the problem in two dimensions lead to a solution of the true MHD system in two dimensions. We will present these in the forthcoming paper [14].

We shall point out that the nonlinear coupling structure in (1.1) is universal and it has been presented in many important models; see the recent survey article [12]. Indeed, the system (1.1) resembles the two-dimensional viscoelastic fluid system:

$$
\begin{cases}
U_t + u\cdot\nabla U = \nabla u U,\\
u_t + u\cdot\nabla u + \nabla p = \Delta u + \nabla\cdot(UU^{\mathsf{T}}),\\
\operatorname{div} u = 0,\\
U|_{t=0} = U_0,\ \ u|_{t=0} = u_0,
\end{cases}
\tag{1.3}
$$

where U denotes the deformation tensor, u is the fluid velocity, and p represents the hydrodynamic pressure (we refer to [13] and the references therein for more details).

In two space dimensions, when $\nabla\cdot U_0 = 0$, it follows from (1.3) that $\nabla\cdot U = 0$ for all $t > 0$. Therefore, one can find a $\phi = (\phi_1, \phi_2)$ such that

$$
U = \begin{pmatrix} -\partial_2\phi_1 & -\partial_2\phi_2\\ \partial_1\phi_1 & \partial_1\phi_2 \end{pmatrix}.
$$

Then (1.3) can be equivalently reformulated as

$$\begin{cases} \phi_t + u \cdot \nabla \phi = 0, \\ u_t + u \cdot \nabla u + \nabla p = \Delta u - \sum_{i=1}^{2} \Delta \phi_i \nabla \phi_i, \\ \operatorname{div} u = 0, \\ \phi|_{t=0} = \phi_0, \ \ u|_{t=0} = u_0. \end{cases} \tag{1.4}$$

One sees the only difference between (1.1) and (1.4) lies in the fact that ϕ is a scalar function in (1.1), while $\phi = (\phi_1, \phi_2)$ is a vector-valued function in (1.4). The authors [13] established the global existence of smooth solutions to the Cauchy problem in the entire space or on a periodic domain for (1.4) in general space dimensions provided that the initial data is sufficiently close to the equilibrium state. The main difficulty in proving this global existence result lies in the free transport equation of ϕ in (1.4), which does not show any dissipative mechanism. However, it is observed in [13] that the combined quantity $w \stackrel{\text{def}}{=} u - (\phi - x)$ satisfies

$$w_t - \Delta w = -u \cdot \nabla w - \nabla p - \sum_{i=1}^{2} \Delta \phi_i \nabla(\phi_i - x_i) + u.$$

This equation for w together with the important fact that $\det\left(\frac{\partial \phi}{\partial x}\right) = 1$ leads to some decay estimates which overcome the difficulty of the hyperbolic nature of the ϕ-equation in (1.4). In fact, the damping mechanism of the system (1.4) can be seen more directly from the linearization of the system ∂_t (1.4):

$$\begin{cases} \phi_{tt} - \Delta \phi - \Delta \phi_t + \nabla q = f, \\ u_{tt} - \Delta u - \Delta u_t + \nabla p = F, \\ \operatorname{div} \phi = \operatorname{div} u = 0. \end{cases} \tag{1.5}$$

For the incompressible MHD equations (1.2), whether there is a dissipation or not is also a very important problem from the physics of plasmas. The heating of high-temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at length scales much larger than the dissipative ones. It has been conjectured that in the MHD systems, with nonvanishing underlying magnetic fields, energy of the system is dispersed and also dissipated at a rate that is independent of the ohmic resistivity [4]. The dispersive nature of the system with nonvanishing magnetic fields was apparently known (from a private communication with Professor A. Majda); see also [16]. In other words, the viscosity (diffusivity) for the magnetic field equation can be 0 yet the whole system may still be dispersive and dissipative when the magnetic fields are not 0. This is the key reason why we consider (1.1) with initial data close to a nontrivial steady state $(x_3, 0)$. Moreover, as the dispersive estimate is often much stronger in three dimensions than in two, as a model problem we shall first consider the global well-posedness of (1.1) with smooth initial data close to $(x_3, 0)$. Of course, x_3 here can be replaced by any nontrivial linear function. On the other hand, without a nonzero magnetic field, the dispersive effect disappears, and in fact it is an open problem to establish global existence of small solutions of (1.1) when the initial data is close to $(0, 0)$.

We note that, after substituting $\phi = x_3 + \psi$ into (1.1), one obtains the following system for (ψ, u) :

$$\begin{cases} \partial_t \psi + u \cdot \nabla \psi + u^3 = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u^h + u \cdot \nabla u^h - \Delta u^h \\ \quad + \nabla_h \partial_3 \psi + \nabla_h p = -\operatorname{div}[\nabla_h \psi \otimes \nabla \psi], \\ \partial_t u^3 + u \cdot \nabla u^3 - \Delta u^3 \\ \quad + (\Delta + \partial_3^2)\psi + \partial_3 p = -\operatorname{div}[\partial_3 \psi \nabla \psi], \\ \operatorname{div} u = 0, \\ (\psi, u)|_{t=0} = (\psi_0, u_0). \end{cases} \tag{1.6}$$

Here and in what follows, we shall always define $u^h := (u^1, u^2)$, $\nabla_h := (\partial_{x_1}, \partial_{x_2})$, and $\Delta_h := \partial_{x_1}^2 + \partial_{x_2}^2$.

Starting from (1.6), a standard energy estimate gives rise to

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla \psi(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2\big) + \|\nabla u(t)\|_{L^2}^2 = 0 \tag{1.7}$$

for smooth enough solutions (ψ, u) of (1.6). The main difficulty in proving the global existence of smooth solutions to (1.6) is thus to find a dissipative mechanism for ψ. Motivated by the heuristic analysis in Section 2.1, we shall employ anisotropic Littlewood-Paley theory to capture the delicate dissipative mechanism of ψ in Section 3. It turns out that the dissipation of u^3 is much stronger than that of u^h, and the horizontal derivatives of ψ, $\nabla_h \psi$, are more dissipative than $\partial_3 \psi$. This, in some sense, also justifies the necessity of using anisotropic Littlewood-Paley theory in Section 3.

Before going further, by taking the divergence of the u-equation of (1.6), we can compute the pressure function p via

$$p = -2\partial_3 \psi + \sum_{i,j=1}^{3} (-\Delta)^{-1}[\partial_i u^j \partial_j u^i + \partial_i \partial_j (\partial_i \psi \partial_j \psi)]. \tag{1.8}$$

Substituting (1.8) into (1.6) results in

$$\begin{cases} \partial_t \psi + u \cdot \nabla \psi + u^3 = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u^h + u \cdot \nabla u^h - \Delta u^h - \nabla_h \partial_3 \psi = f^h, \\ \partial_t u^3 + u \cdot \nabla u^3 - \Delta u^3 + \Delta_h \psi = f^v, \\ \operatorname{div} u = 0, \\ (\psi, u)|_{t=0} = (\psi_0, u_0) \end{cases} \tag{1.9}$$

with f^h and f^v being defined by

$$
\begin{aligned}
f^h &\stackrel{\text{def}}{=} -\sum_{i,j=1}^{3} \nabla_h(-\Delta)^{-1}[\partial_i u^j \partial_j u^i + \partial_i\partial_j(\partial_i\psi\partial_j\psi)] \\
&\quad - \sum_{j=1}^{3} \partial_j(\nabla_h\psi\partial_j\psi), \\
f^v &\stackrel{\text{def}}{=} -\sum_{i,j=1}^{3} \partial_3(-\Delta)^{-1}[\partial_i u^j \partial_j u^i + \partial_i\partial_j(\partial_i\psi\partial_j\psi)] \\
&\quad - \sum_{j=1}^{3} \partial_j(\partial_3\psi\partial_j\psi).
\end{aligned} \tag{1.10}
$$

The object of this paper is to prove the following global well-posedness theorem for (1.9):

THEOREM 1.1. *Let* $s_1 \in (-\frac{1}{2}, 0)$, $s_2 \geq 3$. *Assume the initial data* (ψ_0, u_0) *satisfies* $(\nabla\psi_0, u_0) \in \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3)$ *and*

$$
\|\nabla\psi_0\|_{\dot{H}^{s_1}\cap\dot{H}^{s_2}} + \|u_0\|_{\dot{H}^{s_1}\cap\dot{H}^{s_2}} \leq c_0 \tag{1.11}
$$

for some c_0 *sufficiently small. Then* (1.9) *has a unique global solution* (u, ψ) *(up to a constant for* ψ*) so that* $\nabla\psi \in C([0,\infty); \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3))$ *with* $\nabla_h\psi \in L^2(\mathbb{R}^+; \dot{H}^{1+s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3))$,

$$
u \in C([0,\infty); \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{1+s_1}(\mathbb{R}^3) \cap \dot{H}^{1+s_2}(\mathbb{R}^3))
$$

and with $u^3 \in L^1(\mathbb{R}^+; \mathrm{Lip}(\mathbb{R}^3))$. *Furthermore, there holds*

$$
\begin{aligned}
&\|u\|_{L^\infty_T(\dot{H}^{s_1})} + \|u\|_{L^\infty_T(\dot{H}^{s_2})} \\
&\quad + \|\nabla\psi\|_{L^\infty_T(\dot{H}^{s_1})} + \|\nabla\psi\|_{L^\infty_T(\dot{H}^{s_2})} + \|\nabla u^3\|_{L^1_T(\mathrm{Lip})} \\
&\quad + c(\|\nabla u\|_{L^2_T(\dot{H}^{s_1})} + \|\nabla u\|_{L^2_T(\dot{H}^{s_2})} + \|\nabla_h\psi\|_{L^2_T(\dot{H}^{1+s_1})} + \|\nabla_h\psi\|_{L^2_T(\dot{H}^{s_2})}) \\
&\qquad \leq C(\|u_0\|_{\dot{H}^{s_1}} + \|u_0\|_{\dot{H}^{s_2}} + \|\nabla\psi_0\|_{\dot{H}^{s_1}} + \|\nabla\psi_0\|_{\dot{H}^{s_2}}) \quad \textit{for any } T < \infty.
\end{aligned}
$$

We want to make some preliminary remarks on the above statement.

Remark 1.2. As ψ is a scalar function in (1.9), we cannot apply the ideas and analysis developed in [13, 15] for (1.3) in order to solve (1.9). To find the hidden dissipation in (1.9) may be trickier than the case of the classical isentropic compressible Navier-Stokes system (CNS) as it was first discussed by Danchin [8]. Indeed, Danchin proved that the variation of the density function around a constant state to (CNS) belongs to $L^2(\mathbb{R}^+; B^{d/2}_{2,1}(\mathbb{R}^d))$ and the velocity field is in $L^1(R^+; B^{(d/2)+1}_{2,1}(\mathbb{R}^d))$ provided the initial data is close enough to a constant state, while for our problem here one could only prove that the horizontal derivatives of ψ, $\nabla_h \psi$, are in the space $L^2(\mathbb{R}^+; H^s(\mathbb{R}^3))$, while the third component of the velocity field, u^3, is in the space $L^1(\mathbb{R}^+; \mathrm{Lip}(\mathbb{R}^3))$. Much of the technical difficulties are in getting around these missing estimates.

Remark 1.3. It is also rather technical to explore the delicate mechanism of partial dissipations in (1.9). We settled the difficulties at the end by applying the anisotropic Littlewood-Paley theory. One of the general questions we face is the issue concerning the propagation of anisotropic regularity for the transport equation. We do not know any existing method to handle such questions in general. Here, to overcome this difficulty, we shall introduce a suitable anisotropic Besov-type norm in Definition 2.3, and then establish a priori estimates in such norms for solutions along with the interpolation inequality between this norm and the classical Sobolev norms (see (2.6)). Of course, one may ask if these anisotropic (Sobolev) estimates can also be established by working only in the physical space. We have done various preliminary calculations and estimates, and we found it is rather difficult also if it is at all possible. On the other hand, by working in the phase space, and using anisotropic Littlewood-Paley analysis, various estimates (although they look rather complicated) seem quite natural.

It is interesting to see how the anisotropic Littlewood-Paley theory can be applied to these evolution equations with degenerations of certain ellipticity (parabolicity) in phase variables. One may compare it with classical Hörmander-type operators (hypoellipticity) generated by suitable vector fields in the physical space. It would be interesting to investigate these operators where the vector fields also evolve with time (and sometimes depend nonlinearly on solutions).

Remark 1.4. It is easy to observe from the linearized system (1.13) of (1.9) that ψ satisfies the degenerate damped wave equation

$$\partial_t^2 \psi - \Delta_h \psi - \Delta \partial_t \psi = \partial_t g^0 - \Delta g^0 - g^v.$$

It is well-known that for the wave system, its solutions decay faster in a higher space dimension, which is a crucial fact to prove the global existence of small solutions to nonlinear systems. By using the method introduced in this paper, we can only solve (1.9) in three dimensions (one may check the technical explanation following (1.13)). The two-dimensional result will be presented in [14] by using a different formulation of the system (1.9).

Scheme of the Proof and Organization of the Paper

Let (ψ, u) be a global smooth solution of (1.9); applying a standard energy estimate to (1.9) leads to

$$\begin{aligned}(1.12)\qquad \frac{d}{dt}\Big\{\frac{1}{2}\Big(\|u(t)\|_{\dot H^s}^2 + \|\nabla\psi(t)\|_{\dot H^s}^2 + \frac{1}{4}\|\Delta\psi(t)\|_{\dot H^s}^2\Big) + \frac{1}{4}(u^3 \mid \Delta\psi)_{\dot H^s}\Big\}\\ + \|\nabla u^h\|_{\dot H^s}^2 + \frac{3}{4}\|\nabla u^3\|_{\dot H^s}^2 + \frac{1}{4}\|\nabla_h \nabla\psi\|_{\dot H^s}^2 =\end{aligned}$$

$$
\begin{aligned}
&= -\big((u\cdot\nabla u) \mid u\big)_{\dot H^s} - \big(\nabla\psi \mid \nabla(u\cdot\nabla\psi)\big)_{\dot H^s} - \frac14\big((u\cdot\nabla u^3) \mid \Delta\psi\big)_{\dot H^s}\\
&\quad - \frac14\big(u^3 \mid \Delta(u\cdot\nabla\psi)\big)_{\dot H^s} - \frac14\big(\Delta(u\cdot\nabla\psi) \mid \Delta\psi\big)_{\dot H^s} + \big(f^h \mid u^h\big)_{\dot H^s}\\
&\quad + \left(f^v \mid u^3 + \frac14\Delta\psi\right)_{\dot H^s},
\end{aligned}
$$

where $(a \mid b)_{\dot H^s}$ denotes the standard $\dot H^s(\mathbb{R}^3)$ inner product of a and b. (1.12) shows that $\nabla_h\psi$ is in the space $L^2(\mathbb{R}^+;\dot H^{1+s}(\mathbb{R}^3))$. After a careful check, we would also need to estimate the $L^1(\mathbb{R}^+;\mathrm{Lip}(\mathbb{R}^3))$-norm of u^3 in order to complete the global energy estimate required in (1.12). Here we not only use the estimates for $\nabla_h\psi$ but also apply the fact that $\operatorname{div} u = 0$ in a rather crucial way.

Toward this, we shall first investigate the spectrum properties to the following linearized system of (1.9):

$$
\begin{cases}
\partial_t\psi + u^3 = g^0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\
\partial_t u^h - \Delta u^h - \nabla_h\partial_3\psi = g^h,\\
\partial_t u^3 - \Delta u^3 + \Delta_h\psi = g^v,\\
\operatorname{div} u = 0,\\
(\psi,u)|_{t=0} = (\psi_0,u_0).
\end{cases}
\tag{1.13}
$$

Simple calculation shows that the symbolic matrix of (1.13) has eigenvalues $\lambda_0(\xi)$ and $\lambda_\pm(\xi)$ given by (2.1) that satisfy (2.2). This shows that smooth solutions of (1.13) decay in a very subtle way; moreover, we will have to decompose our frequency analysis into two parts: $\{\xi = (\xi_h,\xi_3) : |\xi|^2 \le 2|\xi_h|\}$ and $\{\xi = (\xi_h,\xi_3) : |\xi|^2 > 2|\xi_h|\}$. It suggests using anisotropic Littlewood-Paley analysis in order to obtain the $L^1(\mathbb{R}^+,\mathrm{Lip}(\mathbb{R}^3))$ estimate of u^3.

Yet due to the fundamental difficulty in propagating anisotropic regularity for the transport equations as we mentioned before, we need to introduce the functional space $\mathcal{B}^{s_1,s_2}$ in Definition 2.3 and to show first that the $L^1(\mathbb{R}^+;\mathcal{B}^{(5/2)-\delta,\delta})$ (for $\delta\in(\frac12,1)$) estimate of u^3 in Proposition 3.8. Note that it would essentially require f^v given by (1.9) to belong to $L^1(\mathbb{R}^+,\mathcal{B}^{(1/2)-\delta,\delta})$. Here we should point out that this requirement is basically equivalent to $f^v\in L^1(\mathbb{R}^+;\mathcal{B}^{(1/2)-\delta}_{2,1}(\mathbb{R}_v)(\mathcal{B}^{\delta}_{2,1}(\mathbb{R}^2_h)))$ for $\delta\in(\frac12,1)$, while for the two-dimensional case, this would require $f^v\in L^1(\mathbb{R}^+;\mathcal{B}^{-\delta}_{2,1}(\mathbb{R}_v)(\mathcal{B}^{\delta}_{2,1}(\mathbb{R}^2_h)))$ for $\delta\in(\frac12,1)$. The latter is impossible due to product laws in Besov spaces in the vertical variable. This is why we will have to use another formulation of (1.1) to prove its global existence of small solutions in the two-dimensional case in our forthcoming paper [14].

In the first part of Section 2, we shall present a heuristic analysis to the linearized system of (1.9), which motivates us to use anisotropic Littlewood-Paley theory below; then we shall collect some basic facts on Littlewood-Paley analysis in Section

2.2. In Section 3, we apply anisotropic Littlewood-Paley theory to explore the dissipative mechanism for a linearized model of (1.9) but with a convection term. In Section 4, we present the proof of Theorem 1.1. Finally, in Appendix A, we present the proofs of Lemmas 3.1, 3.2, and 3.5, and in Appendix B, we present the proof of the lemmas in Section 4.

Let us complete this section by describing the notation we shall use in this paper.

Notation

For A, B two operators, we denote $[A; B] = AB-BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$; $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$. We shall denote by $(a \mid b)$ the $L^2(\mathbb{R}^3)$ inner product of a and b. $(d_{j,k})_{j,k\in\mathbb{Z}}$ (respectively, $(c_j)_{j\in\mathbb{Z}}$) will be a generic element of $\ell^1(\mathbb{Z}^2)$ (respectively, $\ell^2(\mathbb{Z})$) so that $\sum_{j,k\in\mathbb{Z}} d_{j,k} = 1$ (respectively, $\sum_{j\in\mathbb{Z}} c_j^2 = 1$). Finally, we denote by $L^p_T(L^q_h(L^r_v))$ the space $L^p([0,T]; L^q(\mathbb{R}_{x_1}\times\mathbb{R}_{x_2}; L^r(\mathbb{R}_{x_3})))$.

2 Preliminary

2.1 Spectral Analysis of the Linearized System

Before dealing with the full system (1.9), we shall make some heuristic analysis of the linearized system (1.13). Indeed, observe that (1.13) can also be equivalently written as

$$\begin{pmatrix}\psi\\ u\end{pmatrix} = e^{t\mathcal{A}(D)}\begin{pmatrix}\psi_0\\ u_0\end{pmatrix} + \int_0^t e^{(t-s)\mathcal{A}(D)}G(s)\,ds,$$

where

$$G(s) = \begin{pmatrix} g^0\\ g^h\\ g^v\end{pmatrix} \quad\text{and}\quad \mathcal{A}(D) = \begin{pmatrix} 0 & 0 & 0 & -1\\ \partial_1\partial_3 & \Delta & 0 & 0\\ \partial_2\partial_3 & 0 & \Delta & 0\\ -\Delta_h & 0 & 0 & \Delta\end{pmatrix}.$$

The symbolic matrix of the differential operator $\mathcal{A}(D)$ reads

$$A(\xi) = \begin{pmatrix} 0 & 0 & 0 & -1\\ -\xi_1\xi_3 & -|\xi|^2 & 0 & 0\\ -\xi_2\xi_3 & 0 & -|\xi|^2 & 0\\ |\xi_h|^2 & 0 & 0 & -|\xi|^2\end{pmatrix},$$

where $\xi_h \overset{\text{def}}{=} (\xi_1, \xi_2)$. It is easy to calculate that this matrix has three different eigenvalues

$$\lambda_0(\xi) = -|\xi|^2, \quad \lambda_\pm = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4|\xi_h|^2}}{2}. \tag{2.1}$$

In the remainder of this subsection, we always use (ψ, u) to denote a sufficiently smooth solution of (1.13) with a zero source term and $\widehat{a}$ the Fourier transform of the distribution a.

Notice that, corresponding to $\lambda_0(\xi)$ in (2.1), $A(\xi)$ has left characteristic vectors

$$(0, \xi_1, \xi_2, \xi_3) \quad \text{and} \quad (0, 0, |\xi_h|^2, \xi_2\xi_3) \quad \text{or} \quad (0, |\xi_h|^2, 0, \xi_1\xi_3).$$

This suggests that the quantities $\xi\cdot\widehat{u}(t,\xi)$ and $(\xi_1\widehat{u^2}-\xi_2\widehat{u^1})(t,\xi)$ decay like $e^{-t|\xi|^2}$. Thanks to the divergence condition of u in (1.13), $\xi\cdot\widehat{u}(t,\xi)=0$, which however does not provide any additional estimate for u.

On the other hand, corresponding to the eigenvalues $\lambda_\pm$ in (2.1), the matrix $A(\xi)$ has left characteristic vectors

$$\left(|\xi_h|^2, 0, 0, -\frac{|\xi|^2}{2}\left(1+\sqrt{1-\frac{4|\xi_h|^2}{|\xi|^4}}\right)\right), \quad \left(\frac{|\xi|^2}{2}\left(1+\sqrt{1-\frac{4|\xi_h|^2}{|\xi|^4}}\right), 0, 0, -1\right),$$

which suggests that $(-\widehat{\Delta_h\psi}+\widehat{\Delta u^3})(t,\xi)$ decays like $e^{-t|\xi|^2}$, whereas the decay properties of $(\widehat{\Delta\psi}+\widehat{u^3})(t,\xi)$ may be more complicated. Indeed, in the case where $2|\xi_h|\geq|\xi|^2$, $(\widehat{\Delta\psi}+\widehat{u^3})(t,\xi)$ decays just like $e^{-t|\xi|^2/2}$; otherwise the decay property of $(\widehat{\Delta\psi}+\widehat{u^3})(t,\xi)$ varies with directions as

$$\lambda_-(\xi) = -\frac{2|\xi_h|^2}{|\xi|^2\left(1+\sqrt{1-\frac{4|\xi_h|^2}{|\xi|^4}}\right)} \to -1 \quad \text{as } |\xi|\to\infty \tag{2.2}$$

only in the ξ_h-direction. This heuristic analysis shows that the dissipative properties of the solutions to (1.9) may be more complicated than that for the linearized system of the isentropic compressible Navier-Stokes system in [8], and this brief analysis also suggests to us to employ the tool of anisotropic Littlewood-Paley theory, which has been used in the study of the anisotropic incompressible Navier-Stokes equations [5, 6, 9, 10, 17–19] to explore the dissipative properties of (1.9). One may check Section 3 below for the detailed rigorous analysis corresponding to various scenarios.

2.2 Littlewood-Paley Theory

The proof of Theorem 1.1 requires a dyadic decomposition of the Fourier variables, or the Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x\in\mathbb{R}^3$ (see, e.g., [1]). Let $\varphi(\tau)$ and $\chi(\tau)$ be smooth functions such that

$$\operatorname{Supp}\varphi \subset \left\{\tau\in\mathbb{R}\ \middle|\ \frac{3}{4}\leq|\tau|\leq\frac{8}{3}\right\} \quad \text{and} \quad \forall\tau>0,\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau)=1,$$

$$\operatorname{Supp}\chi \subset \left\{\tau\in\mathbb{R}\ \middle|\ |\tau|\leq\frac{4}{3}\right\} \quad \text{and} \quad \chi(\tau)+\sum_{j\geq 0}\varphi(2^{-j}\tau)=1.$$

For $a \in \mathcal{S}'(\mathbb{R}^3)$, we set

$$
\begin{aligned}
&\Delta_k^h a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{a}), \qquad && S_k^h a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\widehat{a}),\\
&\Delta_\ell^v a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), \qquad && S_\ell^v a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}),\\
&\Delta_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), \qquad && S_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}),
\end{aligned} \tag{2.3}
$$

with $\mathcal{F}^{-1}a$ being the inverse Fourier transform of the distribution a. Then the dyadic operators satisfy the property of almost orthogonality:

$$
\Delta_k\Delta_j a \equiv 0 \quad \text{if } |k-j| \geq 2 \quad \text{and} \quad \Delta_k(S_{j-1}a\Delta_j a) \equiv 0 \quad \text{if } |k-j| \geq 5. \tag{2.4}
$$

Similar properties hold for Δ_k^h and Δ_ℓ^v.

We recall now the definition of homogeneous Besov spaces from [1].

DEFINITION 2.1 (Definition 2.15 of [1]). Let $(p,r) \in [1,+\infty]^2$, $s \in \mathbb{R}$. The homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^3)$ consists of those distributions $u \in \mathcal{S}'_h(\mathbb{R}^3)$, which means that $u \in \mathcal{S}'(\mathbb{R}^3)$ and $\lim_{j\to-\infty}\|S_j u\|_{L^\infty} = 0$ (see definition 1.26 of [1]) such that

$$
\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} (2^{qs}\|\Delta_q u\|_{L^p})_{\ell^r(\mathbb{Z})} < \infty.
$$

Remark 2.2.

(1) It is easy to observe that the homogeneous Besov space $\dot{B}^s_{2,2}(\mathbb{R}^3)$ coincides with the classical homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$.

(2) Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$, and $u \in \mathcal{S}'(\mathbb{R}^3)$. Then u belongs to $\dot{B}^s_{p,r}(\mathbb{R}^3)$ if and only if there exists $\{c_{j,r}\}_{j\in\mathbb{Z}}$ such that $\|c_{j,r}\|_{\ell^r} = 1$ and

$$
\|\Delta_j u\|_{L^p} \leq C c_{j,r}\, 2^{-js}\|u\|_{\dot{B}^s_{p,r}} \quad \text{for all } j \in \mathbb{Z}.
$$

To explore the delicate dissipative mechanism of system (1.9), instead of using the classical anisotropic-type Besov spaces as found in [5, 6, 10, 17], we need to introduce the following norm:

DEFINITION 2.3. Let $s_1, s_2 \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^3)$; we define the norm

$$
\|u\|_{\mathcal{B}^{s_1,s_2}} \stackrel{\text{def}}{=} \sum_{j,k\in\mathbb{Z}^2} 2^{js_1}2^{ks_2}\left\|\Delta_j\Delta_k^h u\right\|_{L^2}.
$$

However, due to the difficulty of propagating anisotropic regularities to solutions of transport equations, we need the following imbedding theorem between $\mathcal{B}^{s_1,s_2}(\mathbb{R}^3)$ defined above and the classical homogeneous Sobolev spaces (see [9] for a similar situation).

LEMMA 2.4. *Let s_1, s_2, τ_1, τ_2 be positive numbers with $s_1 < \tau_1 + \tau_2 < s_2$. Then if $a \in \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3)$, $a \in \mathcal{B}^{\tau_1,\tau_2}$, there holds*

$$
\|a\|_{\mathcal{B}^{\tau_1,\tau_2}} \lesssim \|a\|_{\dot{H}^{s_1}} + \|a\|_{\dot{H}^{s_2}}.
$$

PROOF. Indeed, thanks to Definition 2.3, we have

$$\|a\|_{\mathcal{B}^{\tau_1,\tau_2}} = \Big(\sum_{k\le j} + \sum_{j<k}\Big) 2^{j\tau_1}2^{k\tau_2}\big\|\Delta_j\Delta_k^h a\big\|_{L^2}.$$

Notice that since $\tau_2 > 0$, we have

$$\begin{aligned}\sum_{k\le j} 2^{j\tau_1}2^{k\tau_2}\big\|\Delta_j\Delta_k^h a\big\|_{L^2} &\lesssim \sum_{j\in\mathbb{Z}} 2^{j\tau_1}\|\Delta_j a\|_{L^2}\sum_{k\le j}2^{k\tau_2}\\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(\tau_1+\tau_2)}\|\Delta_j a\|_{L^2} \lesssim \|a\|_{\dot B^{\tau_1+\tau_2}_{2,1}}.\end{aligned} \tag{2.5}$$

Along the same line, we get, by using the fact that $j \ge k - N_0$ in the operator $\Delta_j\Delta_k^h$, that

$$\begin{aligned}\sum_{k<j} 2^{j\tau_1}2^{k\tau_2}\big\|\Delta_j\Delta_k^h a\big\|_{L^2} &\lesssim \sum_{k\in\mathbb{Z}} 2^{k\tau_2}\big\|\Delta_k^h a\big\|_{L^2}\sum_{j<k}2^{j\tau_1}\\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{k(\tau_1+\tau_2)}\big\|\Delta_k^h a\big\|_{L^2}\\ &\lesssim \sum_{k\le j-N_0} 2^{k(\tau_1+\tau_2)}\big\|\Delta_j\Delta_k^h a\big\|_{L^2}\\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(\tau_1+\tau_2)}\|\Delta_j a\|_{L^2} \lesssim \|a\|_{\dot B^{\tau_1+\tau_2}_{2,1}},\end{aligned}$$

which together with (2.5) completes the proof of the lemma. □

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin-Lerner type spaces $\widetilde{L}^\lambda_T(B^s_{p,r}(\mathbb{R}^3))$ from [1].

DEFINITION 2.5. Let $s \le \frac{3}{p}$ (or, in general, $s \in \mathbb{R}$), $(r, \lambda, p) \in [1, +\infty]^3$, and $T \in (0, +\infty]$. We define the $\widetilde{L}^\lambda_T(\dot B^s_{p\,r}(\mathbb{R}^3))$-norm by

$$\|f\|_{\widetilde{L}^\lambda_T(\dot B^s_{p,r})} \stackrel{\text{def}}{=} \bigg(\sum_{q\in\mathbb{Z}} 2^{qrs}\Big(\int_0^T \|\Delta_q f(t)\|^\lambda_{L^p}\,dt\Big)^{\frac{r}{\lambda}}\bigg)^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(\dot B^s_{p,r})$.

Remark 2.6. Corresponding to Definitions 2.3 and 2.5, we define the norm

$$\|f\|_{\widetilde{L}^2(\mathcal{B}^{s_1,s_2})} \stackrel{\text{def}}{=} \sum_{j,k\in\mathbb{Z}^2} 2^{js_1}2^{ks_2}\big\|\Delta_j\Delta_k^h u\big\|_{L^2_T(L^2)}.$$

In particular, it follows from the same line of proof for Lemma 2.4 that

$$\|f\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_1,\tau_2})} \lesssim \|f\|_{L^2_T(\dot H^{s_1})} + \|f\|_{L^2_T(\dot H^{s_2})} \tag{2.6}$$

with τ_1, τ_2 and s_1, s_2 being given by Lemma 2.4.

As we shall repeatedly use the anisotropic Littlewood-Paley theory in what follows, for the convenience of the reader, we list some basic facts here.

LEMMA 2.7. *Let $\mathcal{B}_h$ (respectively, $\mathcal{B}_v$) be a ball of $\mathbb{R}^2_h$ (respectively $\mathbb{R}_v$), and $\mathcal{C}_h$ (respectively $\mathcal{C}_v$) a ring of $\mathbb{R}^2_h$ (respectively $\mathbb{R}_v$); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then the following hold:*
If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_h$, then

$$\|\partial^\alpha_{x_h} a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{k(|\alpha|+2(\frac{1}{p_2}-\frac{1}{p_1}))}\|a\|_{L^{p_2}_h(L^{q_1}_v)}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{B}_v$, then

$$\left\|\partial^\beta_3 a\right\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{\ell(\beta+(\frac{1}{q_2}-\frac{1}{q_1}))}\|a\|_{L^{p_1}_h(L^{q_2}_v)}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_h$, then

$$\|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-kN}\sup_{|\alpha|=N}\left\|\partial^\alpha_{x_h} a\right\|_{L^{p_1}_h(L^{q_1}_v)}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{C}_v$, then

$$\|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-\ell N}\left\|\partial^N_3 a\right\|_{L^{p_1}_h(L^{q_1}_v)}.$$

PROOF. Those inequalities are classical (see, for instance, [6, 17]). For the reader's convenience, we shall prove the third one in the specific case when $N = 1$. Let us consider $\widetilde{\varphi}$ in $\mathcal{D}(\mathbb{R}^2\setminus\{0\})$ such that $\widetilde{\varphi}$ has value 1 near $\mathcal{C}_h$. Then for any tempered distribution a such that the support of $\widehat{a}$ is included in $2^k\mathcal{C}_h$, we have

$$\widehat{a} = 2^{-k}i\big(\xi_1\widetilde{\varphi}_1(2^{-k}\xi_h) + \xi_2\widetilde{\varphi}_2(2^{-k}\xi_h)\big)\widehat{a} \quad \text{with } \widetilde{\varphi}_n(\xi_h) \stackrel{\text{def}}{=} -\frac{i\xi_n\widetilde{\varphi}(\xi_h)}{|\xi_h|^2}.$$

Then, we have

$$\begin{aligned} a = 2^{-k}\,\mathrm{div}_h\,\overrightarrow{\Delta}^h_k a, \quad \overrightarrow{\Delta}^h_k a &\stackrel{\text{def}}{=} \big(\Delta^h_{k,1}a, \Delta^h_{k,2}a\big), \\ \Delta^h_{k,n}a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\widetilde{\varphi}_n(2^{-k}\xi_h)\widehat{a}). \end{aligned} \tag{2.7}$$

A similar formula will be useful later on, and it proves the third inequality of the lemma in the particular case when $N = 1$. □

Let us conclude this section by modifying the isotropic paradifferential decomposition of J. M. Bony [2] to the setting of anisotropic version. We first recall the isotropic paradifferential decomposition from [2]: let $a, b \in \mathcal{S}'(\mathbb{R}^3)$,

$$ab = T(a,b) + \mathcal{R}(a,b) \quad \text{or} \quad ab = T(a,b) + \overline{T}(a,b) + R(a,b), \tag{2.8}$$

where

$$T(a,b)=\sum_{j\in\mathbb{Z}}S_{j-1}a\Delta_j b,\quad \bar{T}(a,b)=T(b,a),\quad \mathcal{R}(a,b)=\sum_{j\in\mathbb{Z}}\Delta_j a S_{j+2}b,$$

$$R(a,b)=\sum_{j\in\mathbb{Z}}\Delta_j a\widetilde{\Delta}_j b,\quad \widetilde{\Delta}_j b=\sum_{\ell=j-1}^{j+1}\Delta_\ell a.$$

We shall also use the following anisotropic version of Bony's decomposition for the horizontal variables:

$$ab=T^h(a,b)+\mathcal{R}^h(a,b)\quad\text{or}\quad ab=T^h(a,b)+\bar{T}^h(a,b)+R^h(a,b), \tag{2.9}$$

where

$$T^h(a,b)=\sum_{k\in\mathbb{Z}}S^h_{k-1}a\Delta^h_k b,\quad \bar{T}^h(a,b)=T^h(b,a),$$
$$\mathcal{R}^h(a,b)=\sum_{k\in\mathbb{Z}}\Delta^h_k a S^h_{k+2}b,\quad R^h(a,b)=\sum_{k\in\mathbb{Z}}\Delta^h_k a\widetilde{\Delta}^h_k b,$$
$$\widetilde{\Delta}^h_k b=\sum_{\ell=k-1}^{k+1}\Delta^h_\ell a.$$

Considering the special structure of the functions in $\mathcal{B}^{s_1,s_2}(\mathbb{R}^3)$, we sometimes use both (2.8) and (2.9) simultaneously.

3 Estimate of a Linear System with Convection Terms

Let $f,\psi,u=(u^h,u^3)$ be smooth enough functions satisfying

$$\begin{cases}\partial_t\psi+u\cdot\nabla\psi+u^3=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t u^3+u\cdot\nabla u^3-\Delta u^3+\Delta_h\psi=f,\\ \operatorname{div}u=0,\\ (\psi,u^3)|_{t=0}=(\psi_0,u^3_0).\end{cases} \tag{3.1}$$

The goal of this section is to derive the a priori $L^1(\mathbb{R}^+;\mathrm{Lip}(\mathbb{R}^3))$ estimate of u^3. This system is a sort of linearized model for the ψ and u^3 equation of (1.9), which turns out to be the most difficult part in exploring a dissipative mechanism for (1.9).

According to the heuristic discussions in Section 2.1, when we work on the energy estimate for (3.1), $\Delta\psi$ should be matched with u^3. Furthermore, we need to split the frequency space into two parts, one is $\{\xi : 2|\xi_h|\geq|\xi|^2\}$, where both u^3 and ψ have very good decay properties, the other part is $\{\xi : 2|\xi_h|<|\xi|^2\}$, where the decay rates of u^3 and ψ are not uniform. Motivated by this, we first apply $\Delta_j\Delta_k^h$ to (3.1) and get that

$$(3.2)\qquad \begin{cases} \partial_t\Delta_j\Delta_k^h\psi+\Delta_j\Delta_k^h(u\cdot\nabla\psi)+\Delta_j\Delta_k^h u^3=0, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t\Delta_j\Delta_k^h u^3+\Delta_j\Delta_k^h(u\cdot\nabla u^3)\\ \quad -\Delta\Delta_j\Delta_k^h u^3+\Delta_h\Delta_j\Delta_k^h\psi=\Delta_j\Delta_k^h f. \end{cases}$$

We emphasize that throughout the paper, we always use Δ_h to denote $\partial_{x_1}^2+\partial_{x_2}^2$. Taking the L^2 inner product of the u^3 equation in (3.2) with $\Delta_j\Delta_k^h u^3$ results in

$$(3.3)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta_j\Delta_k^h u^3(t)\|_{L^2}^2+\|\nabla\Delta_j\Delta_k^h u^3(t)\|_{L^2}^2\\ &\quad+\big(\Delta_h\Delta_j\Delta_k^h\psi\mid\Delta_j\Delta_k^h u^3\big)\\ &\qquad=-\big(\Delta_j\Delta_k^h(u\cdot\nabla u^3)\mid\Delta_j\Delta_k^h u^3\big)+\big(\Delta_j\Delta_k^h f\mid\Delta_j\Delta_k^h u^3\big). \end{aligned}$$

However, thanks to the ψ-equation of (3.2) and using integration by parts, we have

$$\begin{aligned} &\big(\Delta_h\Delta_j\Delta_k^h\psi\mid\Delta_j\Delta_k^h u^3\big)=\\ &\qquad\frac{1}{2}\frac{d}{dt}\|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2-\big(\Delta_h\Delta_j\Delta_k^h\psi\mid\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big). \end{aligned}$$

Hence we obtain

$$(3.4)\qquad \begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|\Delta_j\Delta_k^h u^3(t)\|_{L^2}^2+\|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2\big)+\|\nabla\Delta_j\Delta_k^h u^3(t)\|_{L^2}^2\\ &\quad=-\big(\Delta_j\Delta_k^h(u\cdot\nabla u^3)\mid\Delta_j\Delta_k^h u^3\big)-\big(\nabla_h\Delta_j\Delta_k^h\psi\mid\nabla_h\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big)\\ &\qquad+\big(\Delta_j\Delta_k^h f\mid\Delta_j\Delta_k^h u^3\big). \end{aligned}$$

To see decay properties of ψ, we take the L^2 inner product of the u^3-equation in (3.2) with $\Delta\Delta_j\Delta_k^h\psi$ and obtain that

$$\begin{aligned} &\big(\Delta_j\Delta_k^h\partial_t u^3\mid\Delta\Delta_j\Delta_k^h\psi\big)-\big(\Delta\Delta_j\Delta_k^h u^3\mid\Delta\Delta_j\Delta_k^h\psi\big)\\ &\quad+\|\nabla_h\nabla\Delta_j\Delta_k^h\psi\|_{L^2}^2\\ &\qquad=-\big(\Delta_j\Delta_k^h(u\cdot\nabla u^3)\mid\Delta\Delta_j\Delta_k^h\psi\big)+\big(\Delta_j\Delta_k^h f\mid\Delta\Delta_j\Delta_k^h\psi\big). \end{aligned}$$

Again thanks to the ψ-equation of (3.2), one has

$$\begin{aligned} \big(\Delta_j\Delta_k^h\partial_t u^3\mid\Delta\Delta_j\Delta_k^h\psi\big)=&\frac{d}{dt}\big(\Delta_j\Delta_k^h u^3\mid\Delta\Delta_j\Delta_k^h\psi\big)-\big(\Delta_j\Delta_k^h u^3\mid\Delta\Delta_j\Delta_k^h\psi_t\big)\\ =&\frac{d}{dt}\big(\Delta_j\Delta_k^h u^3\mid\Delta\Delta_j\Delta_k^h\psi\big)-\|\nabla\Delta_j\Delta_k^h u^3\|_{L^2}^2\\ &+\big(\Delta_j\Delta_k^h u^3\mid\Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big) \end{aligned}$$

and

$$-\big(\Delta\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h\psi\big) = \frac{1}{2}\frac{d}{dt}\|\Delta\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2 + \big(\Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi) \mid \Delta\Delta_j\Delta_k^h\psi\big),$$

which gives rise to

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}\|\Delta\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2 + \big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h\psi\big)\Big\}\\
&\quad + \|\nabla_h\nabla\Delta_j\Delta_k^h\psi\|_{L^2}^2 - \|\nabla\Delta_j\Delta_k^h u^3\|_{L^2}^2\\
&\quad = -\big(\Delta_j\Delta_k^h(u\cdot\nabla u^3) \mid \Delta\Delta_j\Delta_k^h\psi\big) - \big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big)\\
&\qquad - \big(\Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi) \mid \Delta\Delta_j\Delta_k^h\psi\big) + \big(\Delta_j\Delta_k^h f \mid \Delta\Delta_j\Delta_k^h\psi\big).
\end{aligned} \tag{3.5}$$

Summing up (3.4) with $\frac{1}{4}\times(3.5)$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{2}\big(\|\Delta_j\Delta_k^h u^3(t)\|_{L^2}^2 + \|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2 + \frac{1}{4}\|\Delta\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2\big)\\
&\qquad + \frac{1}{4}\big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h\psi\big)\Big\}\\
&\quad + \frac{3}{4}\|\nabla\Delta_j\Delta_k^h u^3(t)\|_{L^2}^2 + \frac{1}{4}\|\nabla_h\nabla\Delta_j\Delta_k^h\psi\|_{L^2}^2\\
&\quad = -\big(\Delta_j\Delta_k^h(u\cdot\nabla u^3) \mid \Delta_j\Delta_k^h u^3\big) - \big(\nabla_h\Delta_j\Delta_k^h\psi \mid \nabla_h\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big)\\
&\qquad - \frac{1}{4}\big(\Delta_j\Delta_k^h(u\cdot\nabla u^3) \mid \Delta\Delta_j\Delta_k^h\psi\big) - \frac{1}{4}\big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big)\\
&\qquad - \frac{1}{4}\big(\Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi) \mid \Delta\Delta_j\Delta_k^h\psi\big)\\
&\qquad + \Big(\Delta_j\Delta_k^h f \,\Big|\, \Delta_j\Delta_k^h u^3 + \frac{1}{4}\Delta\Delta_j\Delta_k^h\psi\Big).
\end{aligned} \tag{3.6}$$

Notice that

$$\frac{1}{4}\big|\big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h\psi\big)\big| \leq \frac{1}{16}\|\Delta\Delta_j\Delta_k^h\psi\|_{L^2}^2 + \frac{1}{4}\|\Delta_j\Delta_k^h u^3\|_{L^2}^2,$$

one has

$$\begin{aligned}
&\frac{1}{4}\|\Delta_j\Delta_k^h u^3\|_{L^2}^2 + \frac{1}{2}\|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2 + \frac{1}{16}\|\Delta\Delta_j\Delta_k^h\psi\|_{L^2}^2\\
&\quad \leq \frac{1}{2}\big(\|\Delta_j\Delta_k^h u^3(t)\|_{L^2}^2 + \|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2 + \frac{1}{4}\|\Delta\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2\big)\\
&\qquad + \frac{1}{4}\big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h\psi\big)\\
&\quad \leq \frac{3}{4}\|\Delta_j\Delta_k^h u^3\|_{L^2}^2 + \frac{1}{2}\|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2 + \frac{3}{16}\|\Delta\Delta_j\Delta_k^h\psi\|_{L^2}^2.
\end{aligned} \tag{3.7}$$

In what follows, we shall use (3.6) and (3.7) to derive the $L^1(\mathbb{R}^+;\mathrm{Lip}(\mathbb{R}^3))$ estimate for u^3 and the dissipative estimates for ψ. As in the heuristic discussions in Section 2.1, we shall separate the analysis into two regions in the Fourier space. The first part, $\{\xi : 2|\xi_h| \geq |\xi|^2\}$, corresponds to the part with $j \leq (k+1)/2$ in (3.6), and the second part, $\{\xi : 2|\xi_h| < |\xi|^2\}$, corresponds to the part for $j > (k+1)/2$ in (3.6).

3.1 Estimate for the Case $j \leq \frac{k+1}{2}$ in (3.6)

In this case, thanks to Lemma 2.7 and (3.7), one has

$$\begin{aligned} g_{j,k}(t)^2 &\overset{\text{def}}{=} \frac{1}{2}\Big(\big\|\Delta_j\Delta_k^h u^3(t)\big\|_{L^2}^2 + \|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2 + \frac{1}{4}\|\Delta\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2\Big) \\ &\quad + \frac{1}{4}\big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h\psi\big) \\ &\sim \big\|\Delta_j\Delta_k^h u^3(t)\big\|_{L^2}^2 + \|\nabla_h\Delta_j\Delta_k^h\psi(t)\|_{L^2}^2. \end{aligned}$$

Applying Lemma 2.7 once again yields

$$\begin{aligned} &\big\|\nabla\Delta_j\Delta_k^h u^3(t)\big\|_{L^2}^2 + \big\|\nabla\nabla_h\Delta_j\Delta_k^h\psi(t)\big\|_{L^2}^2 \geq \\ &\qquad c2^{2j}\big(\big\|\Delta_j\Delta_k^h u^3(t)\big\|_{L^2}^2 + \big\|\nabla_h\Delta_j\Delta_k^h\psi(t)\big\|_{L^2}^2\big) \geq c2^{2j}g_{j,k}(t)^2. \end{aligned}$$

For any $\varepsilon > 0$, dividing (3.6) by $g_{j,k}(t)+\varepsilon$, then taking $\varepsilon \to 0$ and integrating the resulting equation over $[0,T]$, we obtain

$$\begin{aligned} &\big\|\Delta_j\Delta_k^h u^3\big\|_{L_T^\infty(L^2)} + \big\|\nabla_h\Delta_j\Delta_k^h\psi\big\|_{L_T^\infty(L^2)} \\ &\quad + c2^{2j}\big(\big\|\Delta_j\Delta_k^h u^3\big\|_{L_T^1(L^2)} + \big\|\nabla_h\Delta_j\Delta_k^h\psi\big\|_{L_T^1(L^2)}\big) \\ &\quad \leq \big\|\Delta_j\Delta_k^h u_0^3\big\|_{L^2} + \big\|\nabla_h\Delta_j\Delta_k^h\psi_0\big\|_{L^2} \\ &\qquad + C\int_0^T \big(\big\|\Delta_j\Delta_k^h(u\cdot\nabla u^3)\big\|_{L^2} + \big\|\nabla_h\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big\|_{L^2} \\ &\qquad\qquad + \big\|\Delta_j\Delta_k^h f\big\|_{L^2}\big)\,dt. \end{aligned} \tag{3.8}$$

Here we have used the fact that

$$\begin{aligned} \|\nabla_h\Delta_j\Delta_k^h(u\cdot\nabla\psi)\|_{L^2} &\geq c2^k\|\Delta_j\Delta_k^h(u\cdot\nabla\psi)\|_{L^2} \\ &\geq c2^{2j}\|\Delta_j\Delta_k^h(u\cdot\nabla\psi)\|_{L^2} \geq c\|\Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi)\|_{L^2}, \end{aligned}$$

and

$$\big\|\nabla_h\Delta_j\Delta_k^h\psi\big\|_{L^2} \geq c\|\Delta\Delta_j\Delta_k^h\psi\|_{L^2} \quad \text{for } j \leq \frac{k+1}{2}.$$

Let us now estimate term by term of the last line in (3.8). This will basically depend on the following two technical lemmas, the proofs of which will be postponed to Appendix A.

LEMMA 3.1. *Let $s > 0$ and a, b be sufficiently smooth functions. Then one has*

$$
\begin{aligned}
&\big\|\Delta_j \Delta_k^h(ab)\big\|_{L^1_T(L^2)}\\
&\lesssim d_{j,k}2^{-js}\Big(\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{s,\frac14})}\|b\|_{\widetilde{L}^2_T(\dot{B}^{\frac54}_{2,1})}+\|a\|_{\widetilde{L}^2_T(\dot{B}^{\frac54}_{2,1})}\|b\|_{\widetilde{L}^2_T(\mathcal{B}^{s,\frac14})}\\
&\qquad+\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{\frac12,\frac34})}\|b\|_{L^2_T(\mathcal{B}^{s,\frac14})}+\|a\|_{L^2_T(\mathcal{B}^{s,\frac14})}\|b\|_{\widetilde{L}^2_T(\mathcal{B}^{\frac12,\frac34})}\Big).
\end{aligned}
\tag{3.9}
$$

Here $(d_{j,k})_{j,k\in\mathbb{Z}}$ is a generic element of $\ell^1(\mathbb{Z}^2)$ so that $\sum_{j,k\in\mathbb{Z}} d_{j,k}=1$.

LEMMA 3.2. *Let $s \in (0, \frac34)$, $\delta \in (0,1)$, and $\psi, u = (u^h, u^3)$ be smooth enough functions. Then one has*

$$
\begin{aligned}
&\big\|\Delta_j \Delta_k^h(u\cdot\nabla\psi)\big\|_{L^1_T(L^2)}\\
&\lesssim d_{j,k}2^{-js}2^{-k}\Big\{\|\nabla u\|_{L^2_T(H^1)}\Big(\|\nabla_h\psi\|_{\widetilde{L}^2_T(\dot{B}^{(3/2)}_{2,1})}+\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{\frac12,\frac34})}\\
&\qquad\qquad+\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1-\varepsilon})}+\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1+s,\varepsilon})}\Big)\\
&\qquad+\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(H^1)}\Big\}
\end{aligned}
\tag{3.10}
$$

for some sufficiently small $\varepsilon > 0$.

Remark 3.3. We should point out that the reason that we use the framework of such complicated function spaces $\widetilde{L}^2_T(\mathcal{B}^{s_1,s_1})$ in Lemma 3.1 and Lemma 3.2 is to have $d_{j,k}$ in (3.9) and (3.10) with $\sum_{j,k\in\mathbb{Z}} d_{j,k}=1$, which is crucial to derive the $L^1_T(\mathrm{Lip}(\mathbb{R}^3))$ estimate for u^3 in Proposition 3.8 below.

Applying Lemma 3.1 and Lemma 3.2 to (3.8), we arrive at the following:

PROPOSITION 3.4. *Let $\delta \in (\frac12, 1)$, $\tau_0 \in [\frac32, \frac52-\delta]$, and $\psi, u = (u^h, u^3)$ be sufficiently smooth functions that solve* (3.1). *Then for $j \le (k+1)/2$, there holds*

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h u^3\big\|_{L^\infty_T(L^2)}+\big\|\nabla_h\Delta_j\Delta_k^h\psi\big\|_{L^\infty_T(L^2)}\\
&\quad+c2^{2j}\big(\big\|\Delta_j\Delta_k^h u^3\big\|_{L^1_T(L^2)}+\big\|\nabla_h\Delta_j\Delta_k^h\psi\big\|_{L^1_T(L^2)}\big)\\
&\quad\lesssim d_{j,k}2^{-j(\tau_0-2)}2^{-k\delta}\Big\{\|u_0^3\|_{H^1}+\|\nabla_h\psi_0\|_{H^1}+\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(H^1)}\\
&\qquad+\|\nabla u\|_{L^2_T(H^1)}\big(\|\nabla u\|_{L^2_T(H^1)}+\|\nabla_h\psi\|_{L^2_T(\dot{H}^1)}+\|\nabla_h\psi\|_{L^2_T(\dot{H}^2)}\big)\Big\}\\
&\qquad+\|\Delta_j\Delta_k^h f\|_{L^1_T(L^2)}.
\end{aligned}
$$

PROOF. Indeed, as $\tau_0 \in [\frac32, \frac52-\delta]$, $\delta+\tau_0-2 \in [\delta-\frac12, \frac12]$, we can split it as $\delta+\tau_0-2 = s_1+s_2$ with $s_1, s_2 > 0$. Then since $j \ge k-N_0$ in $\Delta_j\Delta_k^h$, we have

$$
\begin{aligned}
\big\|\Delta_j\Delta_k^h u_0^3\big\|_{L^2} &\lesssim d_{j,k}2^{-js_1}2^{-ks_2}\|u_0^3\|_{\mathcal{B}^{s_1,s_2}}\\
&\lesssim d_{j,k}2^{-j(\tau_0-2)}2^{-k\delta}\|u_0^3\|_{H^1},
\end{aligned}
$$

and a similar estimate is valid for $\nabla_h\psi_0$.

Applying (2.6), Lemma 3.1 for $s = \tau_0 - 1 + \delta \in [\frac{1}{2} + \delta, \frac{3}{2}]$, and Lemma 3.5 for $s = \tau_0 - 2 + \delta \in [\delta - \frac{1}{2}, \frac{1}{2}]$, one has

$$\begin{aligned}\|\Delta_j \Delta_k^h(u\cdot\nabla u^3)\|_{L^1_T(L^2)} &\lesssim 2^j \|\Delta_j\Delta_k^h(uu^3)\|_{L^1_T(L^2)}\\ &\lesssim d_{j,k}2^{-j(\tau_0-2+\delta)}\|\nabla u\|^2_{L^2_T(H^1)},\end{aligned}$$

and

$$\begin{aligned}&\|\nabla_h\Delta_j\Delta_k^h(u\cdot\nabla\psi)\|_{L^1_T(L^2)}\\ &\quad\lesssim d_{j,k}2^{-j(\tau_0-2+\delta)}\big\{\|\nabla u\|_{L^2_T(H^1)}(\|\nabla_h\psi\|_{L^2_T(\dot H^1)} + \|\nabla_h\psi\|_{L^2_T(\dot H^2)})\\ &\qquad + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\|\partial_3\psi\|_{\widetilde L^\infty_T(H^1)}\big\}.\end{aligned}$$

Substituting the above estimates into (3.8) and using once again the fact that $j \geq k - N_0$ in the operator $\Delta_j\Delta_k^h$ leads to Proposition 3.4. □

3.2 Estimate for the Case $j > \frac{k+1}{2}$ in (3.6)

In this case, thanks to Lemma 2.7 and (3.7), one has

$$\begin{aligned}&\frac{1}{2}\big(\|\Delta_j\Delta_k^h u^3(t)\|^2_{L^2} + \|\nabla_h\Delta_j\Delta_k^h\psi(t)\|^2_{L^2} + \frac{1}{4}\|\Delta\Delta_j\Delta_k^h\psi(t)\|^2_{L^2}\big)\\ &\quad + \frac{1}{4}\big(\Delta_j\Delta_k^h u^3 \mid \Delta\Delta_j\Delta_k^h\psi\big)\\ &\quad\sim \|\Delta_j\Delta_k^h u^3\|^2_{L^2} + \|\Delta\Delta_j\Delta_k^h\psi(t)\|^2_{L^2},\end{aligned}$$

which along with (3.6), Lemma 2.7, and a similar derivation of (3.8) ensures that

$$\begin{aligned}&\|\Delta_j\Delta_k^h u^3\|_{L^\infty_T(L^2)} + \|\Delta\Delta_j\Delta_k^h\psi\|_{L^\infty_T(L^2)}\\ &\quad + c\frac{2^{2k}}{2^{2j}}\big(\|\Delta_j\Delta_k^h u^3\|_{L^1_T(L^2)} + \|\Delta\Delta_j\Delta_k^h\psi\|_{L^1_T(L^2)}\big)\\ &\leq \|\Delta_j\Delta_k^h u_0^3\|_{L^2} + \|\Delta\Delta_j\Delta_k^h\psi_0\|_{L^2}\\ &\quad + C\int_0^T \big(\|\Delta_j\Delta_k^h(u\cdot\nabla u^3)\|_{L^2} + \|\Delta\Delta_j\Delta_k^h(u\cdot\nabla\psi)\|_{L^2}\\ &\qquad + \|\Delta_j\Delta_k^h f\|_{L^2}\big)dt.\end{aligned}\tag{3.11}$$

Here we used the fact $j \geq (k+1)/2$ so that $2^{2j} \geq 2^{2(k+1)}/2^{2j}$. Thus the estimate for this part of (ψ, u^3) can be reduced to that for the last line of (3.11). In what follows, for any distribution a, we shall always denote

$$a_{\mathfrak{l}} \stackrel{\text{def}}{=} \sum_{k\geq 2j-1}\Delta_j\Delta_k^h a \quad\text{and}\quad a_{\mathfrak{h}} \stackrel{\text{def}}{=} \sum_{k<2j-1}\Delta_j\Delta_k^h a\tag{3.12}$$

and

$$\|a\|_{\widetilde{L}^1_T(\mathcal{B}^{s_1,s_2}_{\infty,1})} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} 2^{ks_2} \sup_{j\geq k-N_0} 2^{js_1} \|\Delta_j \Delta_k^h a\|_{L^1_T(L^2)} \quad \text{for } s_1, s_2 \in \mathbb{R}\,. \tag{3.13}$$

Here is the key lemma.

LEMMA 3.5. *Let* $\delta \in (\frac{1}{2}, 1)$, $s_0 \in (\frac{1}{2}, \delta)$, *and* $\tau_0 \in [\frac{3}{2}, \frac{5}{2} - \delta]$. *Let* $\psi, u = (u^h, u^3)$ *be sufficiently smooth functions. Then one has*

$$\begin{aligned}&\|\Delta_j \Delta_k^h (u\cdot\nabla\psi)\|_{L^1_T(L^2)}\\ &\quad\lesssim d_{j,k} 2^{-j\tau_0} 2^{-k\delta}\big\{\|\nabla u\|_{L^2_T(H^{\frac{5}{2}})}\big(\|\nabla_h\psi\|_{L^2_T(\dot H^{s_0})} + \|\nabla_h\psi\|_{L^2_T(\dot H^3)}\big)\\ &\qquad\qquad + \big(\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{1,\frac{1}{2}+\delta}_{\infty,1})} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)}\big\}.\end{aligned}$$

We assume this lemma for the time being, the proof of which will be presented in Appendix A.

Remark 3.6. Lemma 3.5 will be the most crucial part used in the proof of Proposition 3.8 below. The main difficulty lies in the estimate of terms like

$$\big(\Delta_j \Delta_k^h (TT^h(u,\nabla\psi)) \,\big|\, \Delta_j \Delta_k^h \psi\big)$$

due to the anisotropic nature in our Littlewood-Paley decompositions. Indeed, comparing with the isotropic case as in (B.4) and (B.10) below, the estimates for commutators of the form $[\Delta_k^h, S_{j-1} S_{k-1}^h u]\cdot\nabla\Delta_j\Delta_k^h\psi$ will lead to estimates involving factors $2^j/2^k$. According to the scaling property of $L^1_T(\mathrm{Lip}(\mathbb{R}^3))$, it would give a $L^1_T(\mathcal{B}^{3/2,1})$ estimate for u^3 provided that the source term f in (3.1) belongs to $L^1_T(\mathcal{B}^{-1/2,1})$. On the other hand, it follows from the proof of Proposition 4.1 that for any smooth enough solution (ψ, u) of (1.9) on $[0,T]$, f^v given by (1.9) belongs only to $\widetilde{L}^1_T(\mathcal{B}^{-1/2,1}_{\infty,1})$. This will lead to a $\widetilde{L}^1_T(\mathcal{B}^{3/2,1}_{\infty,1})$ estimate for u^3 instead, and it does not imply its $L^1_T(\mathrm{Lip}(\mathbb{R}^3))$ estimate. The latter turns out to be the most crucial item in the proof of Theorem 1.1. This also explains why we need to work in the more complicated function spaces $L^1_T(\mathcal{B}^{5/2-\delta,\delta})$ with $\delta\in(\frac{1}{2},1)$ for u^3 in what follows.

With Lemma 3.2, one can deduce, by a similar proof to that for Proposition 3.4 and (3.11), the following:

PROPOSITION 3.7. *Let* $\delta \in (\frac{1}{2}, 1)$, $s_0 \in (\frac{1}{2}, \delta)$, *and* $\tau_0 \in [\frac{3}{2}, \frac{5}{2} - \delta]$. *Let* $\psi, u = (u^h, u^3)$ *be sufficiently smooth functions that solve* (3.1). *Then for* $j > (k+1)/2$

there holds

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h u^3\big\|_{L^\infty_T(L^2)}+\big\|\Delta\Delta_j\Delta_k^h\psi\big\|_{L^\infty_T(L^2)}\\
&\quad+c\big(2^{2(k-j)}\big\|\Delta_j\Delta_k^h u^3\big\|_{L^1_T(L^2)}+2^{2k}\big\|\Delta_j\Delta_k^h\psi\big\|_{L^1_T(L^2)}\big)\\
&\qquad\lesssim\big\|\Delta_j\Delta_k^h f\big\|_{L^1_T(L^2)}+d_{j,k}2^{-j(\tau_0-2)}2^{-k\delta}\Big\{\|u_0^3\|_{H^1}+\|\Delta\psi_0\|_{H^1}\\
&\qquad\quad+\|\nabla u\|_{L^2_T(H^{\frac52})}\big(\|\nabla_h\psi\|_{L^2_T(\dot H^{s_0})}+\|\nabla_h\psi\|_{L^2_T(\dot H^3)}+\|\nabla u\|_{L^2_T(H^1)}\big)\\
&\qquad\quad+\big(\big(\|u^3\|_{\widetilde L^1_T(\mathcal{B}^{1,\frac12+\delta}_{\infty,1})}+\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}\Big\}
\end{aligned}
$$

where $\|\cdot\|_{\widetilde L^1_T(\mathcal{B}^{1,1/2+\delta}_{\infty,1})}$ *is given by* (3.13).

3.3 $L^1_T(\mathrm{Lip}(\mathbb{R}^3))$ Estimate of u^3

The goal of this subsection is to combine Proposition 3.4 and Proposition 3.7 to derive an $L^1_T(\mathrm{Lip}(\mathbb{R}^3))$-estimate of u^3 for solutions of the equation (3.1).

PROPOSITION 3.8. *Under the same assumptions as in Proposition* 3.7, *for any* $\tau_0\in(\frac32,\frac52-\delta]$, *one has*

$$
\begin{aligned}
&\|u^3\|_{\widetilde L^1_T(\mathcal{B}^{\frac32,\delta}_{\infty,1})}+\|\psi_{\mathfrak{l}}\|_{\widetilde L^1_T(\mathcal{B}^{\frac32,1+\delta}_{\infty,1})}+\|\psi_{\mathfrak{h}}\|_{\widetilde L^1_T(\mathcal{B}^{-\frac12,2+\delta}_{\infty,1})}\\
&\quad+\|u^3\|_{L^1_T(\mathcal{B}^{\tau_0,\delta})}+\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{\tau_0,1+\delta})}+\|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{\tau_0-2,2+\delta})}\\
&\qquad\lesssim\|u_0^3\|_{H^1}+\|\nabla\psi_0\|_{H^2}+\|f\|_{L^1_T(\mathcal{B}^{\tau_0-2,\delta})}+\|f\|_{\widetilde L^1_T(\mathcal{B}^{-\frac12,\delta}_{\infty,1})}\\
&\qquad\quad+\|\nabla u\|_{L^2_T(H^{\frac52})}\big(\|\nabla_h\psi\|_{L^2_T(\dot H^{s_0})}+\|\nabla_h\psi\|_{L^2_T(\dot H^3)}+\|\nabla u\|_{L^2_T(H^1)}\big)\\
&\qquad\quad+\big(\|u^3\|_{\widetilde L^1_T(\mathcal{B}^{\frac32,\delta}_{\infty,1})}+\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}
\end{aligned}
\tag{3.14}
$$

with $\psi_{\mathfrak{l}},\psi_{\mathfrak{h}}$ *given by* (3.12) *and* $\|\cdot\|_{\widetilde L^1_T(\mathcal{B}^{1,1/2+\delta}_{\infty,1})}$ *defined by* (3.13).

PROOF. Thanks to (3.13), we have

$$
\begin{aligned}
\|u^3\|_{\widetilde L^1_T(\mathcal{B}^{1,\frac12+\delta}_{\infty,1})}&=\sum_{k\in\mathbb{Z}}2^{k(\frac12+\delta)}\sup_{j\geq k-N_0}2^j\big\|\Delta_j\Delta_k^h u^3\big\|_{L^1_T(L^2)}\\
&\lesssim\sum_{k\in\mathbb{Z}}2^{k\delta}\sup_{j\geq k-N_0}2^{\frac{3j}{2}}\big\|\Delta_j\Delta_k^h u^3\big\|_{L^1_T(L^2)}=\|u^3\|_{\widetilde L^1_T(\mathcal{B}^{3/2,\delta}_{\infty,1})}.
\end{aligned}
$$

Notice that since $\tau_0 \in (\frac{3}{2}, \frac{5}{2}-\delta]$, it follows from Proposition 3.4 and Proposition 3.7 that

$$
\begin{aligned}
&\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{\tau_0,1+\delta})} + \|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{\tau_0-2,2+\delta})} \\
&\quad\lesssim \|u_0^3\|_{H^1} + \|\nabla\psi_0\|_{H^2} + \|f\|_{L^1_T(\mathcal{B}^{\tau_0-2,\delta})} \\
&\qquad + \big(\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{\frac{3}{2},\delta}_{\infty,1})} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)} \\
&\qquad + \|\nabla u\|_{L^2_T(H^{\frac{5}{2}})}\big(\|\nabla_h\psi\|_{L^2_T(\dot{H}^{s_0})} + \|\nabla_h\psi\|_{L^2_T(\dot{H}^3)} + \|\nabla u\|_{L^2_T(H^1)}\big).
\end{aligned}
\tag{3.15}
$$

On the other hand, applying Lemma 2.7 to (3.3) gives

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h u^3\big\|_{L^\infty_T(L^2)} + c2^{2j}\big\|\Delta_j\Delta_k^h u^3\big\|_{L^1_T(L^2)} \\
&\quad\le \big\|\Delta_j\Delta_k^h u_0^3\big\|_{L^2} + C\big(2^{2k}\big\|\Delta_j\Delta_k^h\psi\big\|_{L^1_T(L^2)} \\
&\qquad\qquad + \big\|\Delta_j\Delta_k^h(u\cdot\nabla u^3)\big\|_{L^1_T(L^2)} + \big\|\Delta_j\Delta_k^h f\big\|_{L^1_T(L^2)}\big),
\end{aligned}
$$

which in particular implies that

$$
\begin{aligned}
&2^{j\tau_0}2^{k\delta}\big\|\Delta_j\Delta_k^h u^3\big\|_{L^1_T(L^2)} \\
&\quad\le 2^{j(\tau_0-2)}2^{k\delta}\big\|\Delta_j\Delta_k^h u_0^3\big\|_{L^2} + C2^{j(\tau_0-2)}2^{k(2+\delta)}\|\Delta_j\Delta_k^h\psi\|_{L^1_T(L^2)} \\
&\qquad + C2^{j(\tau_0-2)}2^{k\delta}\big(\big\|\Delta_j\Delta_k^h(u\cdot\nabla u^3)\big\|_{L^1_T(L^2)} + \big\|\Delta_j\Delta_k^h f\big\|_{L^1_T(L^2)}\big).
\end{aligned}
$$

Substituting (3.15) into the above inequality yields

$$
\begin{aligned}
&\sum_{k<2j+1} 2^{j\tau_0}2^{k\delta}\big\|\Delta_j\Delta_k^h u^3\big\|_{L^1_T(L^2)} \\
&\quad\lesssim \|u_0^3\|_{H^1} + \|\nabla\psi_0\|_{H^2} + \|f\|_{L^1_T(\mathcal{B}^{\tau_0-2,\delta})} \\
&\qquad + \big(\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{\frac{3}{2},\delta}_{\infty,1})} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)} \\
&\qquad + \|\nabla u\|_{L^2_T(H^{\frac{5}{2}})}\big(\|\nabla_h\psi\|_{L^2_T(\dot{H}^{s_0})} + \|\nabla_h\psi\|_{L^2_T(\dot{H}^3)} + \|\nabla u\|_{L^2_T(H^1)}\big).
\end{aligned}
$$

From this inequality and Proposition 3.4, we obtain

$$
\begin{aligned}
&\|u^3\|_{L^1_T(\mathcal{B}^{\tau_0,\delta})} \\
&\quad\lesssim \|u_0^3\|_{H^1} + \|\nabla\psi_0\|_{H^2} + \|f\|_{L^1_T(\mathcal{B}^{\tau_0-2,\delta})} \\
&\qquad + \big(\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{\frac{3}{2},\delta}_{\infty,1})} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)} \\
&\qquad + \|\nabla u\|_{L^2_T(H^{\frac{5}{2}})}\big(\|\nabla_h\psi\|_{L^2_T(\dot{H}^{s_0})} + \|\nabla_h\psi\|_{L^2_T(\dot{H}^3)} \\
&\qquad\qquad + \|\nabla u\|_{L^2_T(H^1)}\big).
\end{aligned}
\tag{3.16}
$$

In the same manner, one may deduce that

$$\begin{aligned}
&\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{\frac{3}{2},\delta}_{\infty,1})} + \|\psi_{\mathfrak{l}}\|_{\widetilde{L}^1_T(\mathcal{B}^{\frac{3}{2},1+\delta}_{\infty,1})} + \|\psi_{\mathfrak{h}}\|_{\widetilde{L}^1_T(\mathcal{B}^{-\frac{1}{2},2+\delta}_{\infty,1})} \\
&\quad \lesssim \|u_0^3\|_{H^1} + \|\nabla\psi_0\|_{H^2} \\
&\qquad + \big(\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{\frac{3}{2},\delta}_{\infty,1})} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)} \\
&\qquad + \|\nabla u\|_{L^2_T(H^{\frac{5}{2}})}\big(\|\nabla_h\psi\|_{L^2_T(\dot{H}^{s_0})} \\
&\qquad\qquad + \|\nabla_h\psi\|_{L^2_T(\dot{H}^3)} + \|\nabla u\|_{L^2_T(H^1)}\big) + \|f\|_{\widetilde{L}^1_T(\mathcal{B}^{-\frac{1}{2},\delta}_{\infty,1})}.
\end{aligned}$$

This together with (3.15) and (3.16) finishes the proof of the proposition. □

Remark 3.9. It follows from Remark B.3 below that if (ψ, u) is a smooth solution of (1.9) on $[0, T]$, the source term f^v given by (1.10) belongs only to $\widetilde{L}^1_T(\mathcal{B}^{-1/2,\delta}_{\infty,1})$ no matter how smooth (ψ, u) is. Hence we do not know how to improve the a priori estimate (3.14) for u^3.

4 Proof of Theorem 1.1

It is well known that local existence of smooth solutions to (1.9) basically follows from the a priori estimate for smooth enough solutions of (1.9) (see [16] for instance). Indeed, given smooth initial data (ψ_0, u_0), instead of the smallness condition (1.11), under the assumption that $\|\nabla\psi_0\|_{H^2}$ is sufficiently small, we can deduce from (4.16) that a smooth enough solution (ψ, u) of (1.9) satisfies a local version of the estimate (4.19), which ensures that (1.9) has a local smooth solution (ψ, u) on $[0, T^*)$. The uniqueness of such solutions can be obtained by taking the difference and then by performing an energy estimate. For simplicity, we skip the details here.

The goal of this section is to prove that $T^* = \infty$ provided that (ψ_0, u_0) satisfies (1.11). As a convention in the rest of this section, we shall always denote by (ψ, u) the unique local smooth solution of (1.9) on $[0, T^*)$. We start the proof of Theorem 1.1 by the $L^1_T(\mathrm{Lip}(\mathbb{R}^3))$-estimate of u^3.

4.1 $L^1_T(\mathrm{Lip}(\mathbb{R}^3))$-Estimate of u^3

Thanks to Proposition 3.8, we need to provide a $L^1_T(\mathcal{B}^{(1/2)-\delta,\delta})$-estimate of f^v given by (1.9).

PROPOSITION 4.1. *Let* $\delta \in (\frac{1}{2}, 1)$, (ψ, u) *be the unique local smooth solution of* (1.10) *on* $[0, T^*)$*, and* $\psi_{\mathfrak{l}}, \psi_{\mathfrak{h}}$ *be determined by* (3.12) *with* $\psi_{\mathfrak{l}} \in L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})$

and $\psi_{\mathfrak{h}} \in L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})$. *Let* f^v *be given by* (1.10). *Then there holds*

$$\begin{aligned} &\|f^v\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,\delta})} \\ &\lesssim \|\nabla u\|^2_{L^2_T(H^1)} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^1)} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^2)} \\ &\quad + \|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}\big(\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})} + \|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big), \end{aligned} \tag{4.1}$$

for any $T < T^*$.

PROOF. The proof of this proposition is based on the following two lemmas.

LEMMA 4.2. *Under the assumptions of Proposition* 4.1, *one has*

$$\begin{aligned} &\big\|\Delta_j\Delta^h_k\Delta_h(-\Delta)^{-1}\partial_3(\partial_3\psi)^2\big\|_{L^1_T(L^2)} \\ &\lesssim d_{j,k}2^{-j(\frac12-\delta)}2^{-k\delta}\Big\{\|\nabla_h\psi\|^2_{L^2_T(\dot H^1)} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^2)} \\ &\quad + \|\partial_3\psi\|_{\widetilde L^\infty_T(H^2)}\big(\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})} + \|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big)\Big\}. \end{aligned}$$

LEMMA 4.3. *Under the assumptions of Proposition* 4.1, *one has*

$$\begin{aligned} &\big\|\Delta_j\Delta^h_k(\partial_3\psi\,\partial_h\psi)\big\|_{L^1_T(L^2)} \\ &\lesssim d_{j,k}2^{-j(\frac12-\delta)}2^{-k(1+\delta)}\Big\{\|\nabla_h\psi\|^2_{L^2_T(\dot H^1)} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^2)} \\ &\quad + \|\partial_3\psi\|_{\widetilde L^\infty_T(H^2)}\big(\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})} + \|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big)\Big\}. \end{aligned}$$

Let us proceed to the proof of Proposition 4.1 with Lemma 4.2 and Lemma 4.3, the proofs of which will be presented in Appendix B. We first rewrite f^v as

$$\begin{aligned} f^v &= -\sum_{i,j=1}^{3}\partial_3(-\Delta)^{-1}[\partial_i u^j\partial_j u^i] - \sum_{i\text{ or }j\neq 3}\partial_3(-\Delta)^{-1}\partial_i\partial_j[\partial_i\psi\partial_j\psi] \\ &\quad - \sum_{j=1}^{2}\partial_j(\partial_3\psi\partial_j\psi) - \big(\partial_3^3(-\Delta)^{-1}(\partial_3\psi)^2 + \partial_3(\partial_3\psi)^2\big). \end{aligned} \tag{4.2}$$

As $\operatorname{div} u = 0$, by Lemma 2.4, Lemma 2.7, and Lemma 3.1, one gets

$$\begin{aligned} \Big\|\Delta_j\Delta^h_k\Big(\sum_{i,j=1}^{3}\partial_3(-\Delta)^{-1}[\partial_i u^j\partial_j u^i]\Big)\Big\|_{L^1_T(L^2)} &\lesssim 2^j\,\|\Delta_j\Delta^h_k(u^iu^j)\|_{L^1_T(L^2)} \\ &\lesssim d_{j,k}2^{-\frac{j}{2}}\|\nabla u\|^2_{L^2_T(H^1)}, \end{aligned}$$

whereas Lemma 2.7 and Lemma 4.3 ensure that

$$\begin{aligned}
&\Big\|\Delta_j\Delta_k^h\Big(\sum_{i\text{ or }j\neq 3}\partial_3(-\Delta)^{-1}\partial_i\partial_j[\partial_i\psi\partial_j\psi]-\sum_{j=1}^{2}\partial_j(\partial_3\psi\partial_j\psi)\Big)\Big\|_{L^1_T(L^2)}\\
&\quad\lesssim d_{j,k}2^{-j(\frac{1}{2}-\delta)}2^{-k\delta}\Big\{\|\nabla_h\psi\|^2_{L^2_T(\dot H^1)}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^2)}\\
&\qquad+\|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}\big(\|\psi_{\mathfrak l}\|_{L^1_T(\mathcal B^{(5/2)-\delta,1+\delta})}+\|\psi_{\mathfrak h}\|_{L^1_T(\mathcal B^{(1/2)-\delta,2+\delta})}\big)\Big\}.
\end{aligned}$$

Finally, as $\partial_3^3(-\Delta)^{-1}(\partial_3\psi)^2+\partial_3(\partial_3\psi)^2=-\Delta_h(-\Delta)^{-1}\partial_3(\partial_3\psi)^2$, we get, by applying Lemma 4.2, that

$$\begin{aligned}
&\big\|\Delta_j\Delta_k^h\big[\partial_3^3(-\Delta)^{-1}(\partial_3\psi)^2+\partial_3(\partial_3\psi)^2\big]\big\|_{L^1_T(L^2)}\\
&\quad\lesssim d_{j,k}2^{-j(\frac{1}{2}-\delta)}2^{-k\delta}\Big\{\|\nabla_h\psi\|^2_{L^2_T(\dot H^1)}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^2)}\\
&\qquad+\|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}\big(\|\psi_{\mathfrak l}\|_{L^1_T(\mathcal B^{(5/2)-\delta,1+\delta})}+\|\psi_{\mathfrak h}\|_{L^1_T(\mathcal B^{(1/2)-\delta,2+\delta})}\big)\Big\}.
\end{aligned}$$

This completes the proof of Proposition 4.1. □

Thanks to Proposition 3.8, Proposition 4.1, and (B.2), we obtain the following $L^1_T(\mathrm{Lip}(\mathbb R^3))$-estimate for u^3:

PROPOSITION 4.4. *Under the assumptions of Proposition* 4.1*, if, in addition, we assume that*

$$\|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}\le\overline c_0 \tag{4.3}$$

for some $\overline c_0$ sufficiently small, then for $\delta\in(\frac{1}{2},1)$, there holds

$$\begin{aligned}
&\|\nabla u^3\|_{L^1_T(L^\infty)}\\
&\quad\le C\big\{\|u_0^3\|_{H^1}+\|\nabla\psi_0\|_{H^2}+\|\nabla u\|^2_{L^2_T(H^{\frac{5}{2}})}\\
&\qquad+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_0})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^3)}\big\}\quad\textit{for } s_0\in(\tfrac{1}{2},\delta).
\end{aligned}\tag{4.4}$$

PROOF. Note that $j\ge k-N_0$ for some fixed integer N_0 in the operator $\Delta_j\Delta_k^h$; for $\delta\in(\frac{1}{2},1)$, one deduces from Lemma 2.7 that

$$\begin{aligned}
\|\nabla u^3\|_{L^1_T(L^\infty)}&\lesssim\sum_{j,k\in\mathbb Z^2}2^j\big\|\Delta_j\Delta_k^hu^3\big\|_{L^1_T(L^\infty)}\\
&\lesssim\sum_{j,k\in\mathbb Z^2}2^{\frac{3j}{2}}2^k\big\|\Delta_j\Delta_k^hu^3\big\|_{L^1_T(L^2)}\\
&\lesssim\sum_{j,k\in\mathbb Z^2}2^{j(\frac{5}{2}-\delta)}2^{k\delta}\big\|\Delta_j\Delta_k^hu^3\big\|_{L^1_T(L^2)}\lesssim\|u^3\|_{L^1_T(\mathcal B^{(5/2)-\delta,\delta})},
\end{aligned}$$

whereas for $s_0 \in (\frac{1}{2}, \delta)$, it follows from the classical interpolation inequality in Sobolev spaces that

$$\|\nabla_h\psi\|^2_{L^2_T(\dot H^1)} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^2)} \lesssim \|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_0})} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^3)}.$$

Consequently, thanks to (4.1) and (B.2), we get by taking $\tau_0 = \frac{5}{2} - \delta$ in (3.14) that

$$\begin{aligned}
&\|\nabla u^3\|_{L^1_T(L^\infty)} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})} + \|\psi_\mathfrak{l}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})} + \|\psi_\mathfrak{h}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\\
&\quad + \|u^3\|_{\widetilde L^1_T(\mathcal{B}^{3/2,\delta}_{\infty,1})} + \|\psi_\mathfrak{l}\|_{\widetilde L^1_T(\mathcal{B}^{3/2,1+\delta}_{\infty,1})} + \|\psi_\mathfrak{h}\|_{\widetilde L^1_T(\mathcal{B}^{-1/2,2+\delta}_{\infty,1})}\\
&\qquad \lesssim \|u^3_0\|_{H^1} + \|\nabla\psi_0\|_{H^2} + \|\nabla u\|^2_{L^2_T(H^{\frac{5}{2}})} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_0})} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^3)}\\
&\qquad\quad + \|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}\big(\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})} + \|u^3\|_{\widetilde L^1_T(\mathcal{B}^{3/2,\delta}_{\infty,1})} + \|\psi_\mathfrak{l}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})}\\
&\qquad\quad + \|\psi_\mathfrak{h}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})} + \|\psi_\mathfrak{l}\|_{\widetilde L^1_T(\mathcal{B}^{3/2,1+\delta}_{\infty,1})} + \|\psi_\mathfrak{h}\|_{\widetilde L^1_T(\mathcal{B}^{-1/2,2+\delta}_{\infty,1})}\big).
\end{aligned}$$

The latter and (4.3) lead to

$$\begin{aligned}
&\|\nabla u^3\|_{L^1_T(L^\infty)} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})} + \|\psi_\mathfrak{l}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})} + \|\psi_h\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\\
&\quad + \|u^3\|_{\widetilde L^1_T(\mathcal{B}^{3/2,\delta}_{\infty,1})} + \|\psi_\mathfrak{l}\|_{\widetilde L^1_T(\mathcal{B}^{3/2,1+\delta}_{\infty,1})} + \|\psi_\mathfrak{h}\|_{\widetilde L^1_T(\mathcal{B}^{-1/2,2+\delta}_{\infty,1})}\\
&\qquad \le C\big\{\|u^3_0\|_{H^1} + \|\nabla\psi_0\|_{H^2} + \|\nabla u\|^2_{L^2_T(H^{\frac{5}{2}})}\\
&\qquad\qquad + \|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_0})} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^3)}\big\},
\end{aligned}$$

which in particular yields (4.4). The conclusion of the proposition follows. □

4.2 Dissipative Estimate of $\nabla_h\psi$

The proof of the decay estimates of $\nabla_h\psi$ is based crucially on the following lemmas:

LEMMA 4.5. *Let $\psi, u = (u^h, u^3)$ be the unique local smooth solution of* (1.9) *on $[0, T^*)$. Then for $s > -\frac{3}{2}$ and $T < T^*$, there holds*

$$\begin{aligned}
&\int_0^T |(\partial\Delta_j(u\cdot\nabla\psi) \mid \partial\Delta_j\psi)|dt\\
&\quad\lesssim c_j^2 2^{-2js}\big\{\|\nabla\psi\|^2_{\widetilde L^\infty_T(\dot H^s)}\|\nabla u^3\|_{L^1_T(L^\infty)} + \big(\|\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})} + \|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^s)}\big)\\
&\qquad \times\big(\|u\|^2_{\widetilde L^2_T(\dot B^{3/2}_{2,1})} + \|\nabla_h\psi\|^2_{\widetilde L^2_T(\dot B^{3/2}_{2,1})} + \|\nabla u\|^2_{L^2_T(\dot H^s)} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^{1+s})}\big)\big\}.
\end{aligned}$$

Here and in what follows, we shall always denote by $(c_j)_{j\in\mathbb{Z}}$ a generic element of $\ell^2(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}} c_j^2 = 1$, and ∂ means ∂_{x_1}, ∂_{x_2}, or ∂_{x_3}.

We shall postpone the proof of Lemma 4.5 to Appendix B. Indeed, the proof of Lemma 4.5 also yields the following corollary:

COROLLARY 4.6. *Under the assumptions of Lemma* 4.5, *one has*

$$\begin{aligned}&\int_0^T \big|\big(\partial\Delta_j(u\cdot\nabla\psi)\ \big|\ \partial\Delta_j\psi\big)\big|dt\\&\quad\lesssim c_j^2 2^{-2js}\Big\{\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}^2\|\nabla u^3\|_{L^1_T(L^\infty)}\\&\qquad+\big(\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\big)\\&\qquad\times\big(\|\nabla u\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}+\|\nabla_h\psi\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}+\|\nabla u\|^2_{L^2_T(\dot{H}^s)}+\|\nabla_h\psi\|^2_{L^2_T(\dot{H}^s)}\big)\Big\}.\end{aligned}$$

PROOF. The proof of this corollary follows the same line of argument as that in the proof of Lemma 4.5. We only need to modify the estimates involving $\|\nabla_h\psi\|_{\dot{H}^{1+s}}$. We first use a similar commutator process as in (B.4) to obtain that

$$\begin{aligned}&\int_0^T \big|\big(\partial\Delta_j(T_u\nabla\psi)\ \big|\ \partial\Delta_j\psi\big)\big|dt\\&\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla u^h\|_{L^2_T(L^\infty)}\|\nabla_h\psi\|_{L^2_T(\dot{H}^s)}\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\\&\qquad+\|\nabla u^3\|_{L^1_T(L^\infty)}\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)}\big),\end{aligned}\tag{4.5}$$

while it follows from (B.8) and (B.9) that

$$\begin{aligned}&\int_0^T \big|\big(\partial\Delta_j(T_{\partial_3\psi}u^3)\ \big|\ \partial\Delta_j\psi\big)\big|dt\\&\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\|\nabla_h\psi\|_{L^2_T(L^\infty)}\\&\qquad+\|\partial_3\psi\|_{L^\infty_T(L^\infty)}\|\nabla_h\psi\|_{L^2_T(\dot{H}^s)}\big)\|\nabla u\|_{L^2_T(\dot{H}^s)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \big|\big(\partial\Delta_j(R(u^3,\partial_3\psi))\ \big|\ \partial\Delta_j\psi\big)\big|dt\lesssim\\&\qquad c_j^2 2^{-2js}\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\|\nabla u\|_{L^2_T(\dot{H}^{\frac{3}{2}})}\|\nabla_h\psi\|_{L^2_T(\dot{H}^s)}.\end{aligned}$$

This completes the proof of Corollary 4.4. □

LEMMA 4.7. *Under the assumptions of Lemma* 4.5 *and for* $s > -2$, *one has*

$$\begin{aligned}\int_0^T &\big|\big(\Delta\Delta_j(u\cdot\nabla\psi)\mid\Delta\Delta_j\psi\big)\big|dt\\ &\lesssim c_j^2 2^{-2js}\Big\{\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\|\nabla u^3\|_{L^1_T(L^\infty)}\\ &\quad+\big(\|\partial_3\psi\|_{\widetilde{L}^\infty(\dot{B}^{3/2}_{2,1})}+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\big)\\ &\qquad\times\big(\|\nabla u^h\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}+\|\nabla_h\psi\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}\\ &\qquad\quad+\|\nabla u\|^2_{L^2_T(\dot{H}^{1+s})}+\|\nabla_h\psi\|^2_{\widetilde{L}^2_T(\dot{H}^{1+s})}\big)\Big\}.\end{aligned}$$

We shall present the proof of Lemma 4.7 in Appendix B.

With Lemma 4.5 to Lemma 4.7, we can prove the following proposition concerning the decay estimates of $\nabla_h\psi$.

PROPOSITION 4.8. *Let* $\psi, u = (u^h, u^3)$ *be the unique local smooth solution of* (1.9) *on* $[0, T^*)$. *Then there exists a positive constant* c *such that for any* $s > -\frac{1}{2}$ *and* $T < T^*$, *there holds*

$$\begin{aligned}&\|u^3\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)}+\|\nabla_h\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)}+\|\nabla_h\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\\ &\quad+c\big(\|\nabla u^3\|^2_{L^2_T(\dot{H}^s)}+\|\nabla_h\psi\|^2_{L^2_T(\dot{H}^{1+s})}\big)\\ &\quad\le\|u_0\|^2_{\dot{H}^s}+\|\nabla_h\psi_0\|^2_{\dot{H}^s}+\|\nabla\psi_0\|^2_{\dot{H}^{1+s}}\\ &\quad+C\Big\{\big(\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\big)\|\nabla u^3\|_{L^1_T(L^\infty)}\\ &\quad+\big(\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)}+\|u^3\|_{\widetilde{L}^\infty_T(\dot{H}^s)}+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\big)\\ &\qquad\times\big(\|\nabla u\|^2_{L^2_T(H^2)}+\|\nabla_h\psi\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}+\|\nabla u\|^2_{L^2_T(\dot{H}^s)}\\ &\qquad\quad+\|\nabla u\|^2_{L^2_T(\dot{H}^{1+s})}+\|\nabla_h\psi\|^2_{L^2_T(\dot{H}^{1+s})}\big)\Big\}.\end{aligned}$$

PROOF. From (1.9), we obtain, by a similar derivation of (3.6), that

$$\begin{aligned}&\frac{d}{dt}\Big\{\frac{1}{2}\big(\|\Delta_j u^3(t)\|^2_{L^2}+\|\nabla_h\Delta_j\psi(t)\|^2_{L^2}+\frac{1}{4}\|\Delta\Delta_j\psi(t)\|^2_{L^2}\big)\\ &\qquad+\frac{1}{4}\big(\Delta_j u^3\mid\Delta\Delta_j\psi\big)\Big\}+\frac{3}{4}\|\nabla\Delta_j u^3(t)\|^2_{L^2}+\frac{1}{4}\|\nabla_h\nabla\Delta_j\psi\|^2_{L^2}\\ &\quad=-(\Delta_j(u\cdot\nabla u^3)\mid\Delta_j u^3)-(\nabla_h\Delta_j\psi\mid\nabla_h\Delta_j(u\cdot\nabla\psi))\\ &\qquad-\frac{1}{4}(\Delta_j(u\cdot\nabla u^3)\mid\Delta\Delta_j\psi)-\frac{1}{4}(\Delta_j u^3\mid\Delta\Delta_j(u\cdot\nabla\psi))\\ &\qquad-\frac{1}{4}(\Delta\Delta_j(u\cdot\nabla\psi)\mid\Delta\Delta_j\psi)+\Big(\Delta_j f^v\mid\Delta_j u^3+\frac{1}{4}\Delta\Delta_j\psi\Big).\end{aligned}\tag{4.6}$$

Applying the standard product laws in Besov spaces, we obtain

$$\begin{aligned}\|u\cdot\nabla u^3\|_{L^1_T(\dot H^s)} &\lesssim \|u\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\nabla u\|_{L^2_T(\dot H^s)}\\ \|u\cdot\nabla\psi\|_{L^2_T(\dot H^{1+s})} &\lesssim \|u\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\\ &\quad+\|\nabla\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla u\|_{L^2_T(\dot H^s)}\end{aligned}$$

for $s>-\frac{3}{2}$. We hence deduce

$$\begin{aligned}&\int_0^T\big|\big(\Delta_j(u\cdot\nabla u^3)\ \big|\ \Delta_j u^3\big)\big|dt\\ &\qquad\lesssim c_j^2 2^{-2js}\|u^3\|_{\widetilde L^\infty_T(\dot H^s)}\|u\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\nabla u\|_{L^2_T(\dot H^s)},\\ &\int_0^T\big|\big(\Delta_j(u\cdot\nabla u^3)\ \big|\ \Delta\Delta_j\psi\big)\big|dt\\ &\qquad\lesssim c_j^2 2^{-2js}\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\|u\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\nabla u\|_{L^2_T(\dot H^s)},\\ &\int_0^T\big|\big(\Delta_j u^3\ \big|\ \Delta\Delta_j(u\cdot\nabla\psi)\big)\big|dt\\ &\qquad\lesssim c_j^2 2^{-2js}\|\nabla u^3\|_{L^2_T(\dot H^s)}\\ &\qquad\quad\times\big(\|u\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}+\|\nabla\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla u\|_{L^2_T(\dot H^s)}\big).\end{aligned}\tag{4.7}$$

Next let us turn to the last term in (4.6). By $\operatorname{div} u=0$, one has

$$\begin{aligned}&\sum_{i,m=1}^3\int_0^T\Big|\Big(\Delta_j\partial_3(-\Delta)^{-1}[\partial_i u^m\partial_m u^i]\ \Big|\ \frac{1}{4}\Delta\Delta_j\psi+\Delta_j u^3\Big)\Big|dt\\ &\quad\lesssim\|\Delta_j(u\cdot\nabla u)\|_{L^1_T(L^2)}\big(\|\Delta\Delta_j\psi\|_{L^\infty_T(L^2)}+\|\Delta_j u^3\|_{L^\infty_T(L^2)}\big)\\ &\quad\lesssim c_j^2 2^{-2js}\|u\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\nabla u\|_{L^2_T(\dot H^s)}\big(\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}+\|u^3\|_{\widetilde L^\infty_T(\dot H^s)}\big).\end{aligned}$$

Using integration by parts and product laws in Sobolev spaces, one further deduces that

$$\begin{aligned}&\sum_{i\ \text{or}\ m\neq 3}\int_0^T\big|\big(\Delta_j\partial_3(-\Delta)^{-1}\partial_i\partial_m[\partial_i\psi\partial_m\psi]\ \big|\ \Delta\Delta_j\psi\big)\big|dt\\ &\quad\lesssim c_j^2 2^{-2js}\|\nabla\psi\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\\ &\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\\ &\qquad\qquad+\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\|\nabla_h\psi\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\big)\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}.\end{aligned}$$

The same estimate holds for $\int_0^T |(\Delta_j \operatorname{div}_h(\partial_3\psi\nabla_h\psi) \mid \Delta\Delta_j\psi)|dt$. In the same manner, we have

$$\begin{aligned}
&\int_0^T \big|\big(\Delta_j\partial_3(-\Delta)^{-1}\Delta_h[\partial_3\psi]^2 \mid \Delta\Delta_j\psi\big)\big|dt\\
&\quad= \int_0^T \big|\big(\Delta_j\nabla_h[\partial_3\psi]^2 \mid \Delta_j\nabla_h\partial_3\psi\big)\big|dt\\
&\quad\lesssim c_j^2 2^{-2js}\|\partial_3\psi\nabla_h\partial_3\psi\|_{L^2_T(\dot H^s)}\|\nabla_h\partial_3\psi\|_{L^2_T(\dot H^s)}\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\\
&\qquad\qquad + \|\nabla_h\psi\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\big)\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}.
\end{aligned}$$

Notice that for $s > -\frac{1}{2}$, we have from Bony's decomposition

$$\begin{aligned}
&\|\nabla\psi\nabla\nabla_h\psi\|_{L^2_T(\dot H^{s-1})} + \|\nabla\psi\nabla_h\psi\|_{L^2_T(\dot H^s)} \lesssim\\
&\qquad \|\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})} + \|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^s)}\|\nabla_h\psi\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}.
\end{aligned} \tag{4.8}$$

Consequently, one has

$$\begin{aligned}
&\int_0^T |(\Delta_j(-\Delta)^{-1}\Delta_h\partial_3(\partial_3\psi)^2 \mid \Delta_j u^3)|dt\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\\
&\qquad + \|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^s)}\|\nabla_h\psi\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\big)\|\nabla u^3\|_{L^2_T(\dot H^s)}.
\end{aligned}$$

The same estimate holds for

$$\sum_{i \text{ or } m\neq 3}\int_0^T |(\Delta_j\partial_3(-\Delta)^{-1}\partial_i\partial_m[\partial_i\psi\partial_m\psi] \mid \Delta_j u^3)|dt$$

and $\int_0^T |(\Delta_j \operatorname{div}_h(\partial_3\psi\nabla_h\psi) \mid \Delta_j u^3)|dt$. We thus deduce from (4.2) that

$$\begin{aligned}
&\int_0^T \big|\big(\Delta_j f^v \mid \frac{1}{4}\Delta\Delta_j\psi + \Delta_j u^3\big)\big|dt\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})} + \|\nabla\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})} + \|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^s)}\\
&\qquad\qquad + \|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})} + \|u^3\|_{\widetilde L^\infty_T(\dot H^s)}\big)\\
&\qquad \times\big(\|\nabla_h\psi\|^2_{\widetilde L^2_T(\dot B^{3/2}_{2,1})} + \|u\|^2_{\widetilde L^2_T(\dot B^{3/2}_{2,1})} + \|\nabla_h\psi\|^2_{L^2_T(\dot H^{1+s})} + \|\nabla u\|_{L^2_T(\dot H^s)}\big).
\end{aligned} \tag{4.9}$$

Finally, by Lemma 2.7, Lemma 4.5, Lemma 4.7, (3.7), and (4.6) to (4.9), we conclude that

$$\begin{aligned}
&\|\Delta_j u^3\|_{L^\infty_T(L^2)}^2 + \|\nabla_h \Delta_j \psi\|_{L^\infty_T(L^2)}^2 + \|\Delta\Delta_j \psi\|_{L^\infty_T(L^2)}^2 \\
&\quad + c2^{2j}\big(\|\Delta_j u^3\|_{L^2_T(L^2)}^2 + \|\nabla_h \psi\|_{L^2_T(L^2)}\big) \\
&\qquad \le \|\Delta_j u_0^3\|_{L^2}^2 + \|\nabla_h \Delta_j \psi_0\|_{L^2}^2 + \|\Delta\Delta_j \psi_0\|_{L^2}^2 \\
&\qquad\quad + Cc_j^2 2^{-2js}\Big\{\big(\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)}^2 + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^{1+s})}^2\big)\|\nabla u^3\|_{L^1_T(L^\infty)} \\
&\qquad\quad + \big(\|\psi\|_{\widetilde{L}^\infty_T(\dot B^{3/2}_{2,1})} + \|u^3\|_{\widetilde{L}^\infty_T(\dot H^s)} + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot B^{3/2}_{2,1})} \\
&\qquad\qquad + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)} + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^{1+s})}\big) \\
&\qquad\quad \times \big(\|u\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}^2 + \|\nabla_h\psi\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}^2 + \|\nabla u\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}^2 + \|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}^2 \\
&\qquad\qquad + \|\nabla u\|_{L^2_T(\dot H^s)}^2 + \|\nabla u\|_{L^2_T(\dot H^{1+s})}^2\big)\Big\}.
\end{aligned}$$

Multiplying the above inequality by 2^{2js}, summing up j over $\mathbb{Z}$, and then making use of the fact that

$$\|a\|_{\dot B^{3/2}_{2,1}} \lesssim \|\nabla a\|_{H^1},$$

we can complete the proof of Proposition 4.8. □

Remark 4.9. By using the commutators estimates in the appendix of [8], one can get the following more precise estimate:

$$\begin{aligned}
&\int_0^T \big|\big(\Delta_j(u\cdot\nabla u^3) \mid \Delta\Delta_j\psi\big) + \big(\Delta_j u^3 \mid \Delta\Delta_j(u\cdot\nabla\psi)\big)\big|dt \\
&\quad \lesssim c_j^2 2^{-2js}\big(\|\nabla u\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)} \\
&\qquad\qquad + \|\nabla\psi\|_{\widetilde{L}^\infty(\dot B^{3/2}_{2,1})}\|\nabla u\|_{L^2_T(\dot H^s)}\big)\|\nabla u^3\|_{L^2_T(\dot H^s)}.
\end{aligned}$$

As we shall not use this estimate in this paper, we won't present the details here.

Remark 4.10. Let $f = (f^h, f^v)$ be given by (1.10); we deduce by a similar proof of (4.9) that

$$\begin{aligned}
&\int_0^T |(\Delta_j f \mid \Delta_j u)|dt \\
&\quad \lesssim c_j^2 2^{-2js}\big(\|\psi\|_{\widetilde{L}^\infty_T(\dot B^{3/2}_{2,1})} + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)} + \|u\|_{\widetilde{L}^\infty_T(\dot H^s)}\big) \\
&\qquad \times \big(\|u\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}^2 + \|\nabla_h\psi\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}^2 + \|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}^2 + \|\nabla u\|_{L^2_T(\dot H^s)}^2\big).
\end{aligned} \tag{4.10}$$

On the other hand, instead of (4.8), the standard product laws in Sobolev spaces also ensure that

$$\|\nabla\psi\nabla\nabla_h\psi\|_{L^2_T(\dot H^{s-1})}+\|\nabla\psi\nabla_h\psi\|_{L^2_T(\dot H^{s})}\lesssim \\ \|\nabla\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot H^s)}+\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^s)}\|\nabla_h\psi\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})},$$

which gives rise to

$$\int_0^T |(\Delta_j(-\Delta)^{-1}\Delta_h\partial_3(\partial_3\psi)^2 \mid \Delta_j u^3)|dt \\ \lesssim c_j^2 2^{-2js}\big(\|\nabla\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot H^s)} \\ +\|\nabla_h\psi\|_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^s)}\big)\|\nabla u^3\|_{L^2_T(\dot H^s)}.$$

The same estimate holds for

$$\sum_{i\ \text{or}\ m\neq 3}\int_0^T |(\Delta_j\partial_3(-\Delta)^{-1}\partial_i\partial_m[\partial_i\psi\partial_m\psi] \mid \Delta_j u^3)|dt$$

and $\int_0^T |(\Delta_j\, \mathrm{div}_h(\partial_3\psi\nabla_h\psi) \mid \Delta_j u^3)|dt$. As a consequence, we infer from (4.2) that

$$\int_0^T |(\Delta_j f^v \mid \Delta_j u^3)|dt \\ \lesssim c_j^2 2^{-2js}\big(\|\nabla\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}+\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^s)}+\|u^3\|_{\widetilde L^\infty_T(\dot H^s)}\big) \\ \times\big(\|u\|^2_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}+\|\nabla_h\psi\|^2_{\widetilde L^2_T(\dot B^{3/2}_{2,1})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^s)}+\|\nabla u\|^2_{L^2_T(\dot H^s)}\big). \tag{4.11}$$

It is easy to observe from the definition of f^h given by (1.10) that a similar estimate to (4.11) also holds for $\int_0^T |(\Delta_j f^h \mid \Delta_j u^h)|dt$.

4.3 Proof of Theorem 1.1

With the previous preparations, we are ready to complete the proof of Theorem 1.1. As discussed at the beginning of this section, let ψ, u be the unique local smooth solution of (1.9) on $[0, T^*)$; it suffices to show $T^* = \infty$ provided that there holds (1.11) for some sufficiently small c_0. For this purpose, we show first, by using a similar derivation as that of (3.3) from the u^h-equation of (1.9), that

$$\frac12\frac{d}{dt}\|\Delta_j u^h(t)\|^2_{L^2}+\|\nabla\Delta_j u^h\|^2_{L^2}-(\nabla_h\partial_3\Delta_j\psi \mid \Delta_j u^h)= \\ -(\Delta_j(u\cdot\nabla u^h) \mid \Delta_j u^h)+\big(\Delta_j f^h \bigm| \Delta_j u^h\big).$$

By $\operatorname{div} u = 0$ and the ψ-equation of (1.9), we have

$$\begin{aligned}
-(\nabla_h \partial_3 \Delta_j \psi \mid \Delta_j u^h) &= \big(\partial_3 \Delta_j \psi \mid \Delta_j \operatorname{div}_h u^h\big) \\
&= -(\partial_3 \Delta_j \psi \mid \Delta_j \partial_3 u^3) \\
&= \frac{1}{2}\frac{d}{dt}\|\Delta_j \partial_3 \psi(t)\|_{L^2}^2 + (\Delta_j \partial_3 \psi \mid \Delta_j \partial_3 (u\cdot\nabla\psi)),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Delta_j u^h(t)\|_{L^2}^2 + \|\Delta_j \partial_3 \psi(t)\|_{L^2}^2) + \|\nabla \Delta_j u^h\|_{L^2}^2 = \\
&\quad -(\Delta_j(u\cdot\nabla u^h) \mid \Delta_j u^h) - (\Delta_j \partial_3 \psi \mid \Delta_j \partial_3 (u\cdot\nabla\psi)) + (\Delta_j f^h \mid \Delta_j u^h).
\end{aligned}$$

In the same manner, we deduce from the ψ- and u^3-equations of (1.9) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Delta_j u^3(t)\|_{L^2}^2 + \|\Delta_j \nabla_h \psi(t)\|_{L^2}^2\big) + \|\nabla \Delta_j u^3\|_{L^2}^2 = \\
&\quad -(\Delta_j(u\cdot\nabla u^3) \mid \Delta_j u^3) - (\Delta_j \nabla_h \psi \mid \Delta_j \nabla_h (u\cdot\nabla\psi)) + (\Delta_j f^v \mid \Delta_j u^3).
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\Delta_j u(t)\|_{L^2}^2 + \|\Delta_j \nabla \psi(t)\|_{L^2}^2\big) + \|\nabla \Delta_j u\|_{L^2}^2 = \\
&\quad -(\Delta_j(u\cdot\nabla u) \mid \Delta_j u) - (\Delta_j \nabla \psi \mid \Delta_j \nabla (u\cdot\nabla\psi)) + (\Delta_j f \mid \Delta_j u)
\end{aligned} \tag{4.12}$$

for $f = (f^h, f^v)$ given by (1.10).

Thanks to Lemma 4.5, (4.7), and (4.10), we infer from (4.12) that

$$\begin{aligned}
&\|\Delta_j u\|_{L^\infty_T(L^2)}^2 + \|\Delta_j \nabla\psi\|_{L^\infty_T(L^2)}^2 + \|\Delta_j \nabla u\|_{L^2_T(L^2)}^2 \\
&\quad \le \|\Delta_j u_0\|_{L^2}^2 + \|\nabla\psi_0\|_{L^2}^2 \\
&\qquad + C c_j^2 2^{-2js}\Big\{\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)}^2 \|\nabla u^3\|_{L^1_T(L^\infty)} \\
&\qquad + \big(\|u\|_{\widetilde{L}^\infty_T(\dot H^s)} + \|\psi\|_{\widetilde{L}^\infty_T(\dot B^{3/2}_{2,1})} + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)}\big) \\
&\qquad\quad \times \big(\|u\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}^2 + \|\nabla_h\psi\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}^2 \\
&\qquad\qquad + \|\nabla u\|_{L^2_T(\dot H^s)}^2 + \|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}^2\big)\Big\}
\end{aligned} \tag{4.13}$$

for $s > -\frac{1}{2}$ and $T < T^*$, while it follows from Corollary 4.6, (4.7), and (4.11) that for $s > -\frac{1}{2}$

$$
\begin{aligned}
&\|\Delta_j u\|^2_{L^\infty_T(L^2)} + \|\Delta_j \nabla\psi\|^2_{L^\infty_T(L^2)} + \|\Delta_j \nabla u\|^2_{L^2_T(L^2)} \\
&\quad \leq \|\Delta_j u_0\|^2_{L^2} + \|\nabla\psi_0\|^2_{L^2} \\
&\qquad + Cc_j^2 2^{-2j(1+s)}\Big\{\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\|\nabla u^3\|_{L^1_T(L^\infty)} \\
&\qquad + \big(\|u\|_{\widetilde{L}^\infty_T(\dot{H}^{1+s})} + \|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})} + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\big) \\
&\qquad\quad \times \big(\|u\|^2_{\widetilde{L}^2_T(B^{3/2})} + \|\nabla u\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} + \|\nabla_h\psi\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} \\
&\qquad\qquad + \|\nabla u\|^2_{L^2_T(\dot{H}^{1+s})} + \|\nabla_h\psi\|^2_{L^2_T(\dot{H}^{1+s})}\big)\Big\}.
\end{aligned}
\tag{4.14}
$$

Combining (4.13) with (4.14) and summing up the resulting inequality for j over $\mathbb{Z}$, we arrive at

$$
\begin{aligned}
&\|u\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|u\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})} \\
&\quad + \|\nabla u\|^2_{L^2_T(\dot{H}^s)} + \|\nabla u\|^2_{L^2_T(\dot{H}^{1+s})} \\
&\qquad \leq \|u_0\|^2_{\dot{H}^s} + \|u_0\|^2_{\dot{H}^{1+s}} + \|\nabla\psi_0\|^2_{\dot{H}^s} + \|\nabla\psi_0\|^2_{\dot{H}^{1+s}} \\
&\qquad\quad + C\Big\{\big(\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\big)\|\nabla u^3\|_{L^1_T(L^\infty)} \\
&\qquad\quad + \big(\|\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})} + \|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})} + \|u\|_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|u\|_{\widetilde{L}^\infty_T(\dot{H}^{1+s})} \\
&\qquad\qquad + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\big) \\
&\qquad\quad \times \big(\|u\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} + \|\nabla u\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} + \|\nabla_h\psi\|^2_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} + \|\nabla u\|^2_{L^2_T(\dot{H}^s)} \\
&\qquad\qquad + \|\nabla u\|^2_{L^2_T(\dot{H}^{1+s})} + \|\nabla_h\psi\|^2_{L^2_T(\dot{H}^{1+s})}\big)\Big\}
\end{aligned}
\tag{4.15}
$$

for $s > -\frac{1}{2}$ and $T < T^*$.

For $\overline{c}_0$ sufficiently small, we define

$$
\widetilde{T} \stackrel{\text{def}}{=} \max\{T < T^* : \|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)} \leq \overline{c}_0\}.
$$

Then by (4.4), Proposition 4.8 and a simple interpolation inequalities in Sobolev spaces, we obtain from (4.15) that

$$
\begin{aligned}
&\|u\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|u\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})} \\
&\quad + c\big(\|\nabla u\|^2_{L^2_T(\dot{H}^s)} + \|\nabla u\|^2_{L^2_T(\dot{H}^{1+s})} + \|\nabla_h\psi\|^2_{L^2_T(\dot{H}^{1+s})}\big) \\
&\qquad \leq C\Big\{\|u_0\|^2_{\dot{H}^s} + \|u_0\|^2_{\dot{H}^{1+s}} + \|\nabla\psi_0\|^2_{\dot{H}^s} + \|\nabla\psi_0\|^2_{\dot{H}^{1+s}} \\
&\qquad\quad + \big(\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^s)} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{1+s})}\big)\big(\|u_0^3\|_{H^1} + \|\nabla\psi_0\|_{H^2}\big) +
\end{aligned}
\tag{4.16}
$$

$$+\big(\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)}+\|u\|_{\widetilde{L}^\infty_T(\dot H^s)}+\|u\|_{\widetilde{L}^\infty_T(\dot H^{1+s})}+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)}$$
$$+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^{1+s})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^s)}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{1+s})}\big)$$
$$\times\big(\|\nabla u\|^2_{L^2_T(H^{\frac52})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_0})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^3)}$$
$$+\|\nabla u\|^2_{L^2_T(\dot H^s)}+\|\nabla u\|^2_{L^2_T(\dot H^{1+s})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{1+s})}\big)\Big\}$$

for $s>-\frac12$, $s_0>\frac12$, and $T<\widetilde T$. In particular, we can assume that s_1,s_2 is given by Theorem 1.1. By separately taking $s=s_1$, $s_0=1+s_1>\frac12$, and $s=s_2\ge 3$ in (4.16) and then summing the resulting inequalities, we conclude that

$$\begin{aligned}
&\big(\|u\|^2_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|u\|^2_{\widetilde{L}^\infty_T(\dot H^{s_2})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_2})}\big)(1-Cc_0)\\
&\quad+c\big(\|\nabla u\|^2_{L^2_T(\dot H^{s_1})}+\|\nabla u\|^2_{L^2_T(\dot H^{s_2})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{1+s_1})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_2})}\big)\\
&\quad\le C\Big\{\|u_0\|^2_{\dot H^{s_1}}+\|u_0\|^2_{\dot H^{s_2}}+\|\nabla\psi_0\|^2_{\dot H^{s_1}}+\|\nabla\psi_0\|^2_{\dot H^{s_2}}\\
&\qquad+\big(\|u\|_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|u\|_{\widetilde{L}^\infty_T(\dot H^{s_2})}+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^{s_1})}\\
&\qquad\quad+\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^{s_2})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_2})}\big)\\
&\qquad\times\big(\|\nabla_h\psi\|^2_{L^2_T(\dot H^{1+s_1})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_2})}\\
&\qquad\quad+\|\nabla u\|^2_{L^2_T(\dot H^{s_1})}+\|\nabla u\|^2_{L^2_T(\dot H^{s_2})}\big)\Big\}
\end{aligned}\tag{4.17}$$

for any $T<\widetilde T$ and c_0 being given by (1.11).

Let us denote

$$\begin{aligned}
\overline{T}\overset{\text{def}}{=}\max\big\{T\le\widetilde T:\ &\|u\|^2_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|u\|^2_{\widetilde{L}^\infty_T(\dot H^{s_2})}\\
&+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_2})}\\
&\le 4C\big(\|u_0\|^2_{\dot H^{s_1}}+\|u_0\|^2_{\dot H^{s_2}}+\|\nabla\psi_0\|^2_{\dot H^{s_1}}+\|\nabla\psi_0\|^2_{\dot H^{s_2}}\big)\big\}.
\end{aligned}\tag{4.18}$$

If we take c_0 in (1.11) is so small that $c_0\le\overline{c}_0/4\sqrt{C}$, we would have $\widetilde T=\overline T$. In what follows, we shall prove that $\overline T=T^*=+\infty$ provided that c_0 in (1.11) is sufficiently small. Indeed, if $\overline T<\infty$, (4.17) ensures that

$$\begin{aligned}
&\big(\|u\|^2_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|u\|^2_{\widetilde{L}^\infty_T(\dot H^{s_2})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_1})}+\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^{s_2})}\big)(1-Cc_0)\\
&\quad+(c-6Cc_0)\big(\|\nabla u\|^2_{L^2_T(\dot H^{s_1})}+\|\nabla u\|^2_{L^2_T(\dot H^{s_2})}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{1+s_1})}\\
&\qquad\qquad+\|\nabla_h\psi\|^2_{L^2_T(\dot H^{s_2})}\big)\\
&\quad\le C\big(\|u_0\|^2_{\dot H^{s_1}}+\|u_0\|^2_{\dot H^{s_2}}+\|\nabla\psi_0\|^2_{\dot H^{s_1}}+\|\nabla\psi_0\|^2_{\dot H^{s_2}}\big)\quad\text{for } t\le\overline T.
\end{aligned}$$

In particular, if we take

$$c_0\le\min\Big\{\frac{1}{2C},\frac{c}{12C},\frac{\overline{c}_0}{4\sqrt{C}}\Big\},$$

the above inequality ensures that

$$\begin{aligned}
&\|u\|^2_{\widetilde{L}^\infty_T(\dot{H}^{s_1})} + \|u\|^2_{\widetilde{L}^\infty_T(\dot{H}^{s_2})} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{s_1})} + \|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot{H}^{s_2})}\\
&\quad + c\big(\|\nabla u\|^2_{L^2_T(\dot{H}^{s_1})} + \|\nabla u\|^2_{L^2_T(\dot{H}^{s_2})} + \|\nabla_h\psi\|^2_{L^2_T(\dot{H}^{1+s_1})}\\
&\qquad + \|\nabla_h\psi\|^2_{L^2_T(\dot{H}^{s_2})}\big)\\
&\qquad \le 2C\big(\|u_0\|^2_{\dot{H}^{s_1}} + \|u_0\|^2_{\dot{H}^{s_2}} + \|\nabla\psi_0\|^2_{\dot{H}^{s_1}} + \|\nabla\psi_0\|^2_{\dot{H}^{s_2}}\big) \quad \text{for } t \le \overline{T}.
\end{aligned} \tag{4.19}$$

This contradicts (4.18), and this in turn shows that $\overline{T} = T^* = \infty$. Moreover, it is standard to deduce from (1.9) and (4.19) that $\nabla\psi \in C([0,\infty); \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3))$ and $u \in C([0,\infty); \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3))$ (see [1] for instance). This completes the proof of Theorem 1.1. ■

Appendix A Proofs of Lemmas 3.1, 3.2, and 3.5

In this appendix, we shall present the detailed proofs of the lemmas in Section 3.

PROOF OF LEMMA 3.1. Applying (2.8) and (2.9) we get first that

$$\begin{aligned}
ab &= (T + \mathcal{R})(T^h + R^h + \overline{T}^h)(a,b)\\
&= TT^h(a,b) + TR^h(a,b) + T\overline{T}^h(a,b)\\
&\quad + \mathcal{R}T^h(a,b) + \mathcal{R}R^h(a,b) + \mathcal{R}\overline{T}^h(a,b).
\end{aligned} \tag{A.1}$$

Considering the support properties of the Fourier transform of $TT^h(a,b)$ and using (2.4), we have

$$\big\|\Delta_j\Delta_k^h(TT^h(a,b))\big\|_{L^1_T(L^2)} \lesssim \sum_{\substack{|j'-j|\le 4\\ |k'-k|\le 4}} \big\|S_{j'-1}S^h_{k'-1}a\big\|_{L^2_T(L^\infty)}\big\|\Delta_{j'}\Delta^h_{k'}b\big\|_{L^2_T(L^2)}.$$

Using Lemma 2.7 one has

$$\begin{aligned}
&\big\|S_{j-1}S^h_{k-1}a\big\|_{L^2_T(L^\infty)}\\
&\quad\lesssim \sum_{\substack{j'\le j-2\\ k'\le k-2}}\sum_{\ell\le j'+N_0} 2^{\frac{\ell}{2}}\big\|\Delta_{j'}\Delta^v_\ell\Delta^h_{k'}a\big\|_{L^2_T(L^\infty_h(L^2_v))}\\
&\quad\lesssim \sum_{\substack{j'\le j-2\\ k'\le k-2}} 2^{\frac{j'}{2}}2^{k'}\big\|\Delta_{j'}\Delta^h_{k'}a\big\|_{L^2_T(L^2)}\\
&\quad\lesssim \sum_{\substack{j'\le j-2\\ k'\le k-2}} 2^{\frac{5j'}{4}}2^{\frac{k'}{4}}\big\|\Delta_{j'}\Delta^h_{k'}a\big\|_{L^2_T(L^2)} \lesssim 2^{\frac{k}{4}}\|a\|_{\widetilde{L}^2_T(\dot{B}^{5/4}_{2,1})},
\end{aligned} \tag{A.2}$$

as $j' \ge k' - N_0$ for some fixed integer N_0 in the operator $\Delta_{j'}\Delta^h_{k'}$. The latter and Remark 2.6 lead to

$$\big\|\Delta_j\Delta^h_k(TT^h(a,b))\big\|_{L^1_T(L^2)} \lesssim d_{j,k}2^{-js}\|a\|_{\widetilde{L}^2_T(\dot{B}^{5/4}_{2,1})}\|b\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1/4})}.$$

The same argument leads also to

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h(\mathcal{R}\overline{T}^h(a,b))\big\|_{L^1_T(L^2)}\\
&\quad\lesssim \sum_{\substack{j'\geq j-N_0\\ |k'-k|\leq 4}}\big\|\Delta_{j'}\Delta_{k'}^h a\big\|_{L^2_T(L^2)}\big\|S_{j'+2}S_{k'-1}^h b\big\|_{L^2_T(L^\infty)}\\
&\quad\lesssim d_{j,k}2^{-js}\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1/4})}\|b\|_{\widetilde{L}^2_T(\dot{B}^{5/4}_{2,1})}.
\end{aligned}
$$

By a similar proof of (A.2) , one gets

$$
\big\|S_{j'-1}\Delta_{k'}^h a\big\|_{L^2_T(L^2_h(L^\infty_v))}\lesssim d_{k'}2^{-\frac{3k'}{4}}\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,3/4})},
$$

which along with Lemma 2.7 ensures

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h(TR^h(a,b))\big\|_{L^1_T(L^2)}\\
&\quad\lesssim 2^k\sum_{\substack{|j'-j|\leq 4\\ k'\geq k-N_0}}\big\|S_{j'-1}\Delta_{k'}^h a\big\|_{L^2_T(L^2_h(L^\infty_v))}\big\|\Delta_{j'}\widetilde{\Delta}_{k'}^h b\big\|_{L^2_T(L^2)}\\
&\quad\lesssim 2^k\sum_{\substack{|j'-j|\leq 4\\ k'\geq k-N_0}} d_{j',k'}2^{-j's}2^{-k'}\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,3/4})}\|b\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1/4})}\\
&\quad\lesssim d_{j,k}2^{-js}\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,3/4})}\|b\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1/4})}.
\end{aligned}
$$

The same estimate holds for $\Delta_j\Delta_k^h(T\overline{T}^h(a,b))$. Furthermore, applying Lemma 2.7 once again, one obtains

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h(\mathcal{R}R^h(a,b))\big\|_{L^1_T(L^2)}\\
&\quad\lesssim 2^k\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}}\|\Delta_{j'}\Delta_{k'}^h a\|_{L^2_T(L^2)}\|S_{j'+2}\Delta_{k'}^h b\|_{L^2_T(L^2_h(L^\infty_v))}\\
&\quad\lesssim d_{j,k}2^{-js}\|a\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1/4})}\|b\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,3/4})}.
\end{aligned}
$$

The same estimate holds for $\Delta_j\Delta_k^h(\mathcal{R}T^h(a,b))$. This completes the proof of the Lemma. □

PROOF OF LEMMA 3.2. As for (A.1), we apply (2.8) and (2.9) to obtain first that

$$
u\cdot\nabla\psi = TT^h(u,\nabla\psi)+T\mathcal{R}^h(u,\nabla\psi)+\mathcal{R}T^h(u,\nabla\psi)+\mathcal{R}\mathcal{R}^h(u,\nabla\psi). \tag{A.3}
$$

By the support property of the Fourier transform of $TT^h(u\cdot\nabla\psi)$, for any $\varepsilon>0$ we get by applying Lemma 2.7 and Remark 2.6 that

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h(TT^h(u,\nabla\psi))\big\|_{L^1_T(L^2)}\\
&\quad\lesssim \sum_{\substack{|j'-j|\leq 4\\ |k'-k|\leq 4}}\big\|S_{j'-1}S_{k'-1}^h u\big\|_{L^2_T(L_h^{\frac{2}{\varepsilon}}(L^\infty_v))}\big\|\Delta_{j'}\Delta_{k'}^h\nabla\psi\big\|_{L^2_T(L_h^{\frac{2}{1-\varepsilon}}(L^2_v))}\\
&\quad\lesssim d_{j,k}2^{-js}2^{-k}\|u\|_{\widetilde{L}^2_T(\dot{B}^{\frac{3}{2}-\varepsilon}_{2,1})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1+s,\varepsilon})}.
\end{aligned}
$$

It follows from the same line of argument that

$$\begin{aligned}&\big\|\Delta_j\Delta_k^h(\mathcal{R}T^h(u\cdot\nabla\psi))\big\|_{L^1_T(L^2)}\\&\lesssim\sum_{\substack{j'\geq j-N_0\\|k'-k|\leq 4}}\|\Delta_{j'}S^h_{k'-1}u\|_{L^2_T(L^\infty_h(L^2_v))}\|S_{j'+2}\Delta^h_{k'}\nabla\psi\|_{L^2_T(L^2_h(L^\infty_v))}\\&\lesssim d_{j,k}2^{-js}2^{-k}\|u\|_{\widetilde{L}^2_T(\mathcal{B}^{1+s,1/4})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,3/4})}.\end{aligned}$$

For any $\delta\in(0,1)$, Lemma 2.7 yields

$$\begin{aligned}\|S_{j'-1}\Delta^h_{k'}u^3\|_{L^1_T(L^\infty)}&\lesssim\sum_{\ell\leq j'-2}2^{\frac{3\ell}{2}}\|\Delta_\ell\Delta^h_{k'}u^3\|_{L^1_T(L^2)}\\&\lesssim\sum_{\substack{\ell\leq j'-2\\ \ell\geq k'-N_0}}d_{\ell,k'}2^{-\ell(1-\delta)}2^{-k'\delta}\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\\&\lesssim d_{k'}2^{-k'}\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}.\end{aligned}$$

From the latter and by taking into consideration the time integrabilities of $\nabla_h\psi$ and $\partial_3\psi$, we obtain, for any $\varepsilon>0$, that

$$\begin{aligned}&\big\|\Delta_j\Delta_k^h(T\mathcal{R}^h(u,\nabla\psi))\big\|_{L^1_T(L^2)}\\&\lesssim\sum_{\substack{|j'-j|\leq 4\\k'\geq k-N_0}}\Big(\big\|S_{j'-1}\Delta^h_{k'}u^h\big\|_{L^2_T(L^2_h(L^\infty_v))}\big\|\Delta_{j'}S^h_{k'+2}\nabla_h\psi\big\|_{L^2_T(L^\infty_h(L^2_v))}\\&\qquad+\big\|S_{j'-1}\Delta^h_{k'}u^3\big\|_{L^1_T(L^\infty)}\big\|\Delta_{j'}S^h_{k'+2}\partial_3\psi\big\|_{L^\infty_T(L^2)}\Big)\\&\lesssim d_{j,k}2^{-js}2^{-k}\Big(\|u\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,1+\varepsilon})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1-\varepsilon})}\\&\qquad+\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^s_{2,1})}\Big).\end{aligned}$$

Finally, as $j\geq k-N_0$ in the operator $\Delta_j\Delta^k_k$ and $s\in(0,\frac{3}{2}]$, we deduce from

$$\big\|\Delta_j\Delta^h_k u^3\big\|_{L^1_T(L^2)}\lesssim d_{j,k}2^{-\frac{3j}{2}}2^{-k}\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\quad\forall\delta\in[0,1]$$

that

$$\begin{aligned}&\big\|\Delta_j\Delta_k^h(\mathcal{R}\mathcal{R}^h(u,\nabla\psi))\big\|_{L^1_T(L^2)}\\&\lesssim\sum_{\substack{j'\geq j-N_0\\k'\geq k-N_0}}\Big(\big\|\Delta_{j'}\Delta^h_{k'}u^h\big\|_{L^2_T(L^2)}\big\|S_{j'+2}S^h_{k'+2}\nabla_h\psi\big\|_{L^2_T(L^\infty)}\\&\qquad+\big\|\Delta_{j'}\Delta^h_{k'}u^3\big\|_{L^1_T(L^2)}\big\|S_{j'+2}S^h_{k'+2}\partial_3\psi\big\|_{L^\infty_T(L^\infty)}\Big)\\&\lesssim d_{j,k}2^{-js}2^{-k}\Big(\|u^h\|_{\widetilde{L}^2_T(\mathcal{B}^{s,1})}\|\nabla_h\psi\|_{L^2_T(\dot{B}^{3/2}_{2,1})}\\&\qquad+\|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^s_{2,1})}\Big).\end{aligned}$$

Summing up the above estimates and using (2.6), we complete the proof of Lemma 3.2. □

PROOF OF LEMMA 3.5. Via Bony's decomposition (2.8) and (2.9), we decompose $u\cdot\nabla\psi$ as in (A.3). For $\delta\in(\frac{1}{2},1)$, we split it as $\delta=\frac{1}{2}+\varepsilon_1+\varepsilon_2$ with $\varepsilon_1,\varepsilon_2>0$. Then it follows from the proof of (A.2) that

$$\|S_{j-1}S^h_{k-1}u\|_{L^2_T(L^\infty)}\lesssim 2^{k(\frac{1}{2}-\varepsilon_1)}\|u\|_{\widetilde{L}^2_T(\dot{B}^{1+\varepsilon_1}_{2,1})}.$$

From this and the support properties of the Fourier transform of $TT^h(u,\nabla\psi)$, we deduce that

$$\begin{aligned}\|\Delta_j\Delta^h_k(TT^h(u\cdot\nabla\psi))\|_{L^1_T(L^2)}&\\ \lesssim\sum_{\substack{|j'-j|\leq 4\\ |k'-k|\leq 4}}&\|S_{j'-1}S^h_{k'-1}u\|_{L^2_T(L^\infty)}\|\Delta_{j'}\Delta^h_{k'}\nabla\psi\|_{L^2_T(L^2)}\\ \lesssim d_{j,k}&2^{-j\tau_0}2^{-k\delta}\|u\|_{\widetilde{L}^2_T(\dot{B}^{1+\varepsilon_1}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{\tau_0+1,\varepsilon_2})}.\end{aligned}\tag{A.4}$$

To estimate $T\mathcal{R}^h(u,\nabla\psi)$, we first need to distinguish $\nabla_h\psi$ and $\partial_3\psi$ and then we use their time integrabilities. Let $\varepsilon_0\in(0,(1-\delta)/2)$; it is easy to check that

$$\begin{aligned}&\|\Delta_j\Delta^h_k(T\mathcal{R}^h(u^h,\nabla_h\psi))\|_{L^1_T(L^2)}\lesssim\\ &\sum_{\substack{|j'-j|\leq 4\\ k'\geq k-N_0}}\|S_{j'-1}\Delta^h_{k'}u^h\|_{L^2_T(L^{4/\delta}_h(L^\infty_v))}\|\Delta_{j'}S^h_{k'+2}\nabla_h\psi\|_{L^2_T(L^{4/(2-\delta)}_h(L^2_v)))}.\end{aligned}$$

Applying Lemma 2.7, we get

$$\begin{aligned}\|S_{j'-1}&\Delta^h_{k'}u^h\|_{L^2_T(L^{4/\delta}_h(L^\infty_v))}\\ &\lesssim\sum_{\substack{\ell\leq j'-2\\ \ell\geq k'-N_0}}2^{\frac{\ell}{2}}2^{k'(1-\frac{\delta}{2})}\|\Delta_\ell\Delta^h_{k'}u^h\|_{L^2_T(L^2)}\\ &\lesssim\sum_{\ell\leq j'-2}2^{\ell(1+\varepsilon_0)}2^{k'(\frac{1-\delta}{2}-\varepsilon_0)}\|\Delta_\ell\Delta^h_{k'}u^h\|_{L^2_T(L^2)}\\ &\lesssim d_{k'}\|u^h\|_{\widetilde{L}^2_T(\mathcal{B}^{1+\varepsilon_0,(1-\delta)/2-\varepsilon_0})}.\end{aligned}$$

A similar argument yields

$$\|\Delta_{j'}S^h_{k'+2}\nabla_h\psi\|_{L^2_T(L^{4/(2-\delta)}_h(L^2_v)))}\lesssim d_{j'}2^{-j'(\tau_0+\delta)}\|\nabla_h\psi\|_{\widetilde{L}^2(\mathcal{B}^{\tau_0+\delta,\delta/2})}.$$

This leads to

$$\begin{aligned}&\|\Delta_j\Delta^h_k(T\mathcal{R}^h(u^h,\nabla_h\psi))\|_{L^1_T(L^2)}\lesssim\\ &\qquad d_{j,k}2^{-j(\tau_0+\delta)}\|u^h\|_{\widetilde{L}^2_T(\mathcal{B}^{1+\varepsilon_0,(1-\delta)/2-\varepsilon_0})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_0+\delta,\delta/2})},\end{aligned}\tag{A.5}$$

whereas Lemma 2.7 and (3.13) lead to

$$\begin{aligned}
&\left\|S_{j'-1}\Delta_{k'}^h u^3\right\|_{L^1_T(L^\infty)}\\
&\lesssim \sum_{\ell\le k'} 2^{\frac{3\ell}{2}}\left\|\Delta_\ell\Delta_{k'}^h u^3\right\|_{L^1_T(L^2)} + \sum_{k'\le\ell\le j'-2} 2^{\frac{\ell}{2}}2^{k'}\left\|\Delta_\ell\Delta_{k'}^h u^3\right\|_{L^1_T(L^2)}\\
&\lesssim d_{k'}2^{-k'\delta}\Big\{2^{-\frac{k'}{2}}\sum_{\ell\le k'}2^{\frac{\ell}{2}} + 2^{\frac{k'}{2}}\sum_{k'\le\ell}2^{-\frac{\ell}{2}}\Big\}\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{1,(1/2)+\delta}_{\infty,1})}\\
&\lesssim d_{k'}2^{-k'\delta}\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{1,(1/2)+\delta}_{\infty,1})}.
\end{aligned}$$

The latter implies

$$\begin{aligned}
&\left\|\Delta_j\Delta_k^h(T\overline{T}^h(u^3,\partial_3\psi))\right\|_{L^1_T(L^2)}\\
&\lesssim \sum_{\substack{|j'-j|\le4\\|k'-k|\le4}}\left\|S_{j'-1}\Delta_{k'}^h u^3\right\|_{L^1_T(L^\infty)}\left\|\Delta_{j'}S_{k'-1}^h\partial_3\psi\right\|_{L^\infty_T(L^2)}\\
&\lesssim \sum_{\substack{|j'-j|\le4\\|k'-k|\le4}} d_{j'}d_{k'}2^{-j'\tau_0}2^{-k'\delta}\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{1,(1/2)+\delta}_{\infty,1})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{\tau_0}_{2,1})}\\
&\lesssim d_{j,k}2^{-j\tau_0}2^{-k\delta}\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{1,(1/2)+\delta}_{\infty,1})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{\tau_0}_{2,1})}.
\end{aligned}$$

We then deduce, by a similar argument as for (A.4), that

$$\begin{aligned}
&\left\|\Delta_j\Delta_k^h(TR^h(u^3,\partial_3\psi))\right\|_{L^1_T(L^2)}\\
&\lesssim \sum_{\substack{|j'-j|\le4\\k'\ge k-N_0}}\left\|S_{j'-1}\Delta_{k'}^h u^3\right\|_{L^2_T(L^\infty)}\left\|\Delta_{j'}\widetilde{\Delta}_{k'}^h\partial_3\psi\right\|_{L^2_T(L^2)}\\
&\lesssim d_{j,k}2^{-j\tau_0}2^{-k\delta}\|u^3\|_{\widetilde{L}^2_T(\dot{B}^{1+\varepsilon_1}_{2,1})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_0+1,\varepsilon_2})}
\end{aligned}$$

for the same $\varepsilon_1,\varepsilon_2$ as in (A.4). As a consequence, we obtain

$$\begin{aligned}
&\left\|\Delta_j\Delta_k^h(T\mathcal{R}^h(u^3,\partial_3\psi))\right\|_{L^1_T(L^2)}\\
&\le \left\|\Delta_j\Delta_k^h(T\overline{T}^h(u^3,\partial_3\psi))\right\|_{L^1_T(L^2)} + \left\|\Delta_j\Delta_k^h(TR^h(u^3,\partial_3\psi))\right\|_{L^1_T(L^2)}\\
&\lesssim d_{j,k}2^{-j\tau_0}2^{-k\delta}\big\{\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{1,(1/2)+\delta}_{\infty,1})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{\tau_0}_{2,1})}\\
&\qquad + \|u^3\|_{\widetilde{L}^2_T(\dot{B}^{1+\varepsilon_1}_{2,1})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_0+1,\varepsilon_2})}\big\}.
\end{aligned} \tag{A.6}$$

Next, let $\varepsilon_3\in(0,\frac{1}{2})$ and apply Lemma 2.7. We obtain

$$\begin{aligned}
&\left\|S_{j'+2}\Delta_{k'}^h\nabla\psi\right\|_{L^2_T(L_h^{\frac{2}{1-\varepsilon_3}}(L^\infty_v))}\\
&\lesssim 2^j\sum_{\ell\le j'+1}2^{\frac{\ell}{2}}2^{k'\varepsilon_3}\left\|\Delta_\ell\Delta_{k'}^h\psi\right\|_{L^2_T(L^2)} \lesssim
\end{aligned}$$

$$\lesssim 2^j 2^{k'(\frac{1}{2}-\delta+\varepsilon_3)} \sum_{\ell \leq j'+1} 2^{\frac{\ell}{2}} 2^{k'(\delta-\frac{1}{2})} \big\|\Delta_\ell \Delta_{k'}^h \psi\big\|_{L^2_T(L^2)}$$
$$\lesssim d_{k'} 2^j 2^{-k'(\frac{1}{2}+\delta-\varepsilon_3)} \|\nabla_h \psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,\delta-\frac{1}{2}})}$$

and

$$\big\|\Delta_{j'} S^h_{k'-1} u\big\|_{L^2_T(L_h^{\frac{2}{\varepsilon_3}}(L^2_v))} \lesssim d_{j'} 2^{-j'(\tau_0+1)} 2^{k'(\frac{1}{2}-\varepsilon_3)} \|u\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_0+1,\frac{1}{2}})},$$

which gives rise to

$$\begin{aligned}
&\big\|\Delta_j \Delta_k^h(\mathcal{R}T^h(u,\nabla\psi))\big\|_{L^1_T(L^2)} \\
&\quad\lesssim \sum_{\substack{j'\geq j-N_0\\ |k'-k|\leq 4}} \big\|\Delta_{j'} S^h_{k'-1} u\big\|_{L^2_T(L_h^{2/\varepsilon_3}(L^2_v))} \big\|S_{j'+2}\Delta^h_{k'}\nabla\psi\big\|_{L^2_T(L_h^{2/(1-\varepsilon_3)}(L^\infty_v))} \\
&\quad\lesssim d_{j,k} 2^{-j\tau_0} 2^{-k\delta} \|u\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_0+1,\frac{1}{2}})} \|\nabla_h\psi\|_{\widetilde{L}^2(\mathcal{B}^{1/2,\delta-\frac{1}{2}})}.
\end{aligned} \tag{A.7}$$

Finally, by the support properties of the Fourier transform of $\mathcal{R}\mathcal{R}^h(u\cdot\nabla\psi)$ and the time integrability of $\nabla_h\psi$ and $\partial_3\psi$, we obtain

$$\begin{aligned}
&\big\|\Delta_j \Delta_k^h(\mathcal{R}\mathcal{R}^h(u,\nabla\psi))\big\|_{L^1_T(L^2)} \\
&\quad\lesssim \sum_{\substack{j'\geq -N_0\\ k'\geq k-N_0}} \Big\{\big\|\Delta_{j'}\Delta^h_{k'} u^h\big\|_{L^2_T(L^2)} \big\|S_{j'+2}S^h_{k'+2}\nabla_h\psi\big\|_{L^2_T(L^\infty)} \\
&\qquad\qquad + \big\|\Delta_{j'}\Delta^h_{k'} u^3\big\|_{L^1_T(L^2)} \big\|S_{j'+2}S^h_{k'+2}\partial_3\psi\big\|_{L^\infty_T(L^\infty)}\Big\} \\
&\quad\lesssim d_{j,k} 2^{-j\tau_0} 2^{-k\delta}\big(\|u^h\|_{\widetilde{L}^2_T(\mathcal{B}^{\tau_0,\delta})}\|\nabla_h\psi\|_{L^2_T(L^\infty)} \\
&\qquad\qquad + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{\tau_0+\delta-1}_{2,1})}\big),
\end{aligned} \tag{A.8}$$

where we used the fact that $\tau_0 \in [\frac{3}{2}, \frac{5}{2}-\delta]$ so that $\tau_0+\delta-1 \leq \frac{3}{2}$ and

$$\big\|S_{j'+2}S^h_{k'+2}\partial_3\psi\big\|_{L^\infty_T(L^\infty)} \lesssim 2^{j(\frac{5}{2}-\delta-\tau_0)}\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{\tau_0+\delta-1}_{2,1})}.$$

Combining (A.4)–(A.8), we arrive at

$$\begin{aligned}
&\big\|\Delta_j\Delta_k^h(u\cdot\nabla\psi)\big\|_{L^1_T(L^2)} \\
&\quad\lesssim d_{j,k}2^{-j\tau_0}2^{-k\delta}\Big\{\|\nabla u\|_{L^2_T(H^{\frac{5}{2}})}\big(\|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{1/2,\delta-\frac{1}{2}})} + \|\nabla_h\psi\|_{L^2_T(\dot{B}^{3/2}_{2,1})} \\
&\qquad + \|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{\tau_0+\delta,\frac{\delta}{2}})} + \|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{\tau_0+1,\varepsilon_2})}\big) \\
&\qquad + \big(\|u^3\|_{\widetilde{L}^1_T(\mathcal{B}^{1,(1/2)+\delta}_{\infty,1})} + \|u^3\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,\delta})}\big)\|\nabla\psi\|_{\widetilde{L}^\infty_T(H^2)}\Big\}.
\end{aligned}$$

From the latter and (2.6), we conclude the proof of the lemma. □

Appendix B Proofs of Lemmas 4.2, 4.3, 4.5, and 4.7

The proof of Proposition 4.1 is complete provided that we present the proof of Lemma 4.2 and Lemma 4.3, which we give now.

PROOF OF LEMMA 4.2. By applying Bony's decomposition (2.8) and (2.9), we see that

$$
\begin{aligned}
\partial_3\psi\nabla_h\partial_3\psi &= TT^h(\partial_3\psi,\nabla_h\partial_3\psi)+T\mathcal{R}^h(\partial_3\psi,\nabla_h\partial_3\psi)\\
&\quad + RT^h(\partial_3\psi,\nabla_h\partial_3\psi)+R\mathcal{R}^h(\partial_3\psi,\nabla_h\partial_3\psi)\\
&\quad + \bar{T}T^h(\partial_3\psi,\nabla_h\partial_3\psi)+\bar{T}\mathcal{R}^h(\partial_3\psi,\nabla_h\partial_3\psi).
\end{aligned}\tag{B.1}
$$

Considering the support property of the Fourier transform of $TT^h(\partial_3\psi,\nabla_h\partial_3\psi)$ and taking into account the different regularity assumptions of $\psi_{\mathfrak{l}}$ and $\psi_{\mathfrak{h}}$, we get, by applying Lemma 2.7, that

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h\,\mathrm{div}_h(-\Delta)^{-1}\partial_3(RT^h(\partial_3\psi,\nabla_h\partial_3\psi))\big\|_{L^1_T(L^2)}\\
&\lesssim 2^{-\frac{j}{2}}2^k\sum_{\substack{j'\geq j-N_0\\|k'-k|\leq 4}}\big\|\Delta_{j'}S^h_{k'-1}\partial_3\psi\big\|_{L^\infty_T(L^\infty_h(L^2_v))}\\
&\qquad\qquad\times\big(\big\|\widetilde{\Delta}_{j'}\Delta^h_{k'}\nabla_h\partial_3\psi_{\mathfrak{l}}\big\|_{L^1_T(L^2)}+\big\|\widetilde{\Delta}_{j'}\Delta^h_{k'}\nabla_h\partial_3\psi_{\mathfrak{h}}\big\|_{L^1_T(L^2)}\big).
\end{aligned}
$$

From this and Definition 2.3, we conclude

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h\,\mathrm{div}_h(-\Delta)^{-1}\partial_3(RT^h(\partial_3\psi,\nabla_h\partial_3\psi))\big\|_{L^1_T(L^2)}\\
&\lesssim 2^{\frac{j}{2}}\sum_{\substack{j'\geq j-N_0\\|k'-k|\leq 4}}d_{j',k'}2^{j'(1-\delta)}2^{-k'\delta}\big(\|\partial_3\psi\|_{\widetilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})}\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})}\\
&\qquad\qquad+\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big)\\
&\lesssim d_{j,k}2^{-j(\frac{1}{2}-\delta)}2^{-k\delta}\big(\|\partial_3\psi\|_{\widetilde{L}^\infty(\dot{B}^{1/2}_{2,1})}\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})}\\
&\qquad\qquad+\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big).
\end{aligned}
$$

Following the same line of argument, we have

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h\,\mathrm{div}_h(-\Delta)^{-1}\partial_3(TT^h(\partial_3\psi,\nabla_h\partial_3\psi))\big\|_{L^1_T(L^2)}\\
&\lesssim 2^{-j}2^k\sum_{\substack{|j'-j|\leq 4\\|k'-k|\leq 4}}\big\|S_{j'-1}S^h_{k'-1}\partial_3\psi\big\|_{L^\infty_T(L^\infty)}\big\|\Delta_{j'}\Delta^h_{k'}\nabla_h\partial_3\psi\big\|_{L^1_T(L^2)}\\
&\lesssim d_{j,k}2^{-j(\frac{1}{2}-\delta)}2^{-k\delta}\big(\|\partial_3\psi\|_{\widetilde{L}^\infty(\dot{B}^{1/2}_{2,1})}\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})}\\
&\qquad\qquad+\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\Delta_j\Delta_k^h\,\mathrm{div}_h(-\Delta)^{-1}\partial_3(\bar{T}T^h(\partial_3\psi,\nabla_h\partial_3\psi))\big\|_{L^1_T(L^2)}\\
&\quad\lesssim\sum_{\substack{|j'-j|\leq 4\\|k'-k|\leq 4}}\|\Delta_{j'}S^h_{k'-1}\partial_3\psi\|_{L^\infty_T(L^2)}\big\|S_{j'-1}\Delta^h_{k'}\nabla_h\partial_3\psi\big\|_{L^1_T(L^\infty)}\lesssim
\end{aligned}
$$

$$\lesssim d_{j,k}2^{-j(\frac{1}{2}-\delta)}2^{-k\delta}\big(\|\partial_3\psi\|_{\widetilde{L}^\infty(\dot{B}^{\frac{1}{2}}_{2,1})}\|\psi_\mathfrak{l}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})}$$
$$+\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\psi_\mathfrak{h}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big).$$

Next, by applying Lemma 2.7 once again and using the fact that $\delta\in(\frac{1}{2},1)$, one gets

$$\big\|\Delta_j\Delta_k^h\operatorname{div}_h(-\Delta)^{-1}\partial_3(R\mathcal{R}^h(\partial_3\psi,\nabla_h\partial_3\psi))\big\|_{L^1_T(L^2)}$$
$$\lesssim 2^{-\frac{j}{2}}2^k\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}}\big\|\Delta_{j'}\Delta_{k'}^h\partial_3\psi\big\|_{L^2_T(L^2)}\big\|\widetilde{\Delta}_{j'}S^h_{k'+2}\nabla_h\partial_3\psi\big\|_{L^2_T(L^\infty_h(L^2_v))}$$
$$\lesssim 2^{-\frac{j}{2}}2^k\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}}2^{-k'}\big\|\Delta_{j'}\Delta_{k'}^h\nabla_h\partial_3\psi\big\|_{L^2_T(L^2)}\big\|\widetilde{\Delta}_{j'}S^h_{k'+2}\nabla_h\partial_3\psi\big\|_{L^2_T(L^\infty_h(L^2_v))},$$

which along with Definition 2.3 ensures that

$$\big\|\Delta_j\Delta_k^h\operatorname{div}_h(-\Delta)^{-1}\partial_3(R\mathcal{R}^h(\partial_3\psi,\nabla_h\partial_3\psi))\big\|_{L^1_T(L^2)}$$
$$\lesssim 2^{\frac{j}{2}}\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}}d_{j',k'}2^{-j'(1-\delta)}2^{-k'\delta}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{2-\delta,\delta/2})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1,\delta/2})}$$
$$\lesssim d_{j,k}2^{-j(\frac{1}{2}-\delta)}2^{-k\delta}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{2-\delta,\delta/2})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1,\delta/2})}.$$

Similarly, we have

$$\big\|\Delta_j\Delta_k^h\operatorname{div}_h(-\Delta)^{-1}\partial_3(T\mathcal{R}^h(\partial_3\psi,\nabla_h\partial_3\psi))\big\|_{L^1_T(L^2)}$$
$$\lesssim\sum_{\substack{|j'-j|\leq 4\\ k'\geq k-N_0}}\big\|S_{j'-1}\Delta_{k'}^h\partial_3\psi\big\|_{L^2_T(L^2_h(L^\infty_v))}\big\|\Delta_{j'}S^h_{k'+2}\nabla_h\partial_3\psi\big\|_{L^2_T(L^\infty_h(L^2_v))}$$
$$\lesssim d_{j,k}2^{-j(\frac{1}{2}-\delta)}2^{-k\delta}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{2-\delta,\delta/2})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1,\delta/2})}.$$

The same estimate holds for $\operatorname{div}_h(-\Delta)^{-1}\partial_3(\overline{T}\mathcal{R}^h(\partial_3\psi,\nabla_h\partial_3\psi))$. We thus conclude the proof via Lemma 2.4. □

Remark B.1. It is easy to see from the proof of Lemma 4.2, under the assumptions of Proposition 4.1, that

$$\big\|\Delta_j\Delta_k^h\Delta_h(-\Delta)^{-1}\partial_3(\partial_3\psi)^2\big\|_{L^1_T(L^2)}$$
$$\lesssim c_k^2 2^{\frac{j}{2}}2^{-k\delta}\big\{\big(\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}\|\psi_\mathfrak{l}\|_{\widetilde{L}^1_T(\mathcal{B}^{3/2,1+\delta}_{\infty,1})}$$
$$+\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\psi_\mathfrak{h}\|_{\widetilde{L}^1_T(\mathcal{B}^{-1/2,2+\delta}_{\infty,1})}\big)+\|\nabla_h\psi\|^2_{\widetilde{L}^2_T(\mathcal{B}^{1,\delta/2})}\big\}.$$

PROOF OF LEMMA 4.3. Similarly to (B.1), we apply Bony's decomposition (2.8) and (2.9) to get that

$$\partial_3\psi\partial_h\psi=TT^h(\partial_3\psi,\partial_h\psi)+T\mathcal{R}^h(\partial_3\psi,\partial_h\psi)+RT^h(\partial_3\psi,\partial_h\psi)$$
$$+R\mathcal{R}^h(\partial_3\psi,\partial_h\psi)+\overline{T}T^h(\partial_3\psi,\partial_h\psi)+\overline{T}\mathcal{R}^h(\partial_3\psi,\partial_h\psi).$$

Considering the different regularity assumptions of $\psi_{\mathfrak{l}}$ and $\psi_{\mathfrak{h}}$ given by (3.12) and applying Lemma 2.7, we obtain

$$\begin{aligned}
&\|\Delta_j\Delta_k^h(RT^h(\partial_3\psi,\partial_h\psi))\|_{L^1_T(L^2)}\\
&\lesssim 2^{\frac{j}{2}}\sum_{\substack{j'\geq j-N_0\\|k'-k|\leq 4}}\|\Delta_{j'}S^h_{k'-1}\partial_3\psi\|_{L^\infty_T(L^\infty_h(L^2_v))}\big(\|\widetilde{\Delta}_{j'}\Delta^h_{k'}\partial_h\psi_{\mathfrak{l}}\|_{L^1_T(L^2)}\\
&\qquad\qquad\qquad\qquad+\|\widetilde{\Delta}_{j'}\Delta^h_{k'}\partial_h\psi_{\mathfrak{h}}\|_{L^1_T(L^2)}\big)\\
&\lesssim d_{j,k}2^{-j(\frac12-\delta)}2^{-k(1+\delta)}\big(\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac12}_{2,1})}\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})}\\
&\qquad\qquad\qquad\qquad+\|\partial_3\psi_h\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\psi\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big).
\end{aligned}$$

In the same way, we can get

$$\begin{aligned}
&\|\Delta_j\Delta_k^h(\overline{T}T^h(\partial_3\psi,\partial_h\psi))\|_{L^1_T(L^2)}\\
&\lesssim\sum_{\substack{|j'-j|\leq 4\\|k'-k|\leq 4}}\big(\|\Delta_{j'}S^h_{k'-1}\partial_3\psi\|_{L^\infty_T(L^2)}\|S_{j'-1}\Delta^h_{k'}\partial_h\psi_{\mathfrak{l}}\|_{L^1_T(L^\infty)}\\
&\qquad\qquad+\|\Delta_{j'}S^h_{k'-1}\partial_3\psi\|_{L^\infty_T(L^\infty)}\|S_{j'-1}\Delta^h_{k'}\partial_h\psi_{\mathfrak{h}}\|_{L^1_T(L^2)}\big)\\
&\lesssim d_{j,k}2^{-j(\frac12-\delta)}2^{-k(1+\delta)}\big(\|\partial_3\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(5/2)-\delta,1+\delta})}\\
&\qquad\qquad\qquad\qquad+\|\partial_3\psi_h\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\psi\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big).
\end{aligned}$$

The same estimate holds for $TT^h(\partial_3\psi,\partial_h\psi)$.

Next, by Lemma 2.7, we obtain

$$\begin{aligned}
&\|\Delta_j\Delta_k^h(R\mathcal{R}^h(\partial_3\psi,\partial_h\psi))\|_{L^1_T(L^2)}\\
&\lesssim 2^{\frac{j}{2}}\sum_{\substack{j'\geq j-N_0\\k'\geq k-N_0}}2^j2^{-k}\|\Delta_{j'}\Delta^h_{k'}\nabla_h\psi\|_{L^1_T(L^2))}\|\widetilde{\Delta}_{j'}S^h_{k'+2}\partial_h\psi\|_{L^\infty_T(L^\infty_h(L^2_v))}\\
&\lesssim d_{j,k}2^{-j(\frac12-\delta)}2^{-k(1+\delta)}\|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{(3/2)-\delta,\delta})}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,1})}.
\end{aligned}$$

It follows from the same line of the proof that

$$\begin{aligned}
&\|\Delta_j\Delta_k^h(T\mathcal{R}^h(\partial_3\psi,\partial_h\psi))\|_{L^1_T(L^2)}\\
&\lesssim\sum_{\substack{|j'-j|\leq 4\\k'\geq k-N_0}}2^{j'}2^{-k'}\|S_{j'-1}\Delta^h_{k'}\nabla_h\psi\|_{L^2_T(L^2_h(L^\infty_v)))}\|\Delta_{j'}S^h_{k'+2}\partial_h\psi\|_{L^2_T(L^\infty_h(L^2_v))}\\
&\lesssim d_{j,k}2^{-j(\frac12-\delta)}2^{-k(1+\delta)}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\mathcal{B}^{1/2,1})}\|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{(3/2)-\delta,\delta})}.
\end{aligned}$$

One derives the same estimate for the term $\overline{T}\mathcal{R}^h(\partial_3\psi,\partial_h\psi))$.

We consequently arrive at

$$\begin{aligned}
&\big\|\Delta_j\Delta_k^h(\partial_3\psi,\partial_h\psi)\big\|_{L^1_T(L^2)}\\
&\quad\lesssim d_{j,k}2^{-j(\frac12-\delta)}2^{-k(1+\delta)}\big\{\|\nabla_h\psi\|_{\widetilde L^2_T(\mathcal{B}^{(3/2)-\delta,\delta})}\|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{1/2,1})}\\
&\qquad+\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot B^{1/2}_{2,1})}\|\psi_{\mathfrak{l}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}+\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\psi_{\mathfrak{h}}\|_{L^1_T(\mathcal{B}^{(1/2)-\delta,2+\delta})}\big\},
\end{aligned}$$

which along with Lemma 2.4 leads to Lemma 4.3. □

Remark B.2. By a similar proof to that for Lemma 4.3, we have

$$\begin{aligned}
&\big\|\Delta_j\Delta_k^h(\partial_3\psi\partial_h\psi)\big\|_{L^1_T(L^2)}\\
&\quad\lesssim d_k2^{\frac{j}{2}}2^{-k(1+\delta)}\{\|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{1/2,\delta})}\|\nabla_h\psi\|_{L^2_T(\mathcal{B}^{1/2,1})}\\
&\qquad\qquad+\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot B^{1/2}_{2,1})}\|\psi_{\mathfrak{l}}\|_{\widetilde L^1_T(\mathcal{B}^{3/2,1+\delta}_{\infty,1})}\\
&\qquad\qquad+\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot B^{3/2}_{2,1})}\|\psi_{\mathfrak{h}}\|_{\widetilde L^1_T(\mathcal{B}^{-1/2,2+\delta}_{\infty,1})}\big\}.
\end{aligned}$$

Remark B.3. With Remarks B.1 and B.2 , we can conclude from the proof of Proposition 4.1 that

$$\begin{aligned}
\|f^v\|_{\widetilde L^1_T(\mathcal{B}^{-1/2,\delta}_{\infty,1})}&\lesssim\|\nabla u\|^2_{L^2_T(H^1)}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^1)}+\|\nabla_h\psi\|^2_{L^2_T(\dot H^2)}\\
&\quad+\|\nabla\psi\|_{\widetilde L^\infty_T(H^2)}\big(\|\psi_{\mathfrak{l}}\|_{\widetilde L^1_T(\mathcal{B}^{3/2,1+\delta}_{\infty,1})}+\|\psi_{\mathfrak{h}}\|_{\widetilde L^1_T(\mathcal{B}^{-1/2,2+\delta}_{\infty,1})}\big)
\end{aligned}\tag{B.2}$$

for any $T<T^*$.

Finally, let us turn to the proof of Lemma 4.5 and Lemma 4.7, which have been used in the proof of Proposition 4.8.

PROOF OF LEMMA 4.5. Using Bony's decomposition (2.8), we see that

$$u\cdot\nabla\psi=T_u\nabla\psi+T_{\nabla\psi}u+R(u,\nabla\psi).$$

We shall deal with $u^h\cdot\nabla_h\psi$ and $u^3\partial_3\psi$ separately using various time integrabilities of $\nabla_h\psi$ and $\partial_3\psi$. First, one observes that

$$\begin{aligned}
\|\Delta_j(T_{u^h}\nabla_h\psi)\|_{L^1_T(L^2)}&\lesssim\sum_{|j'-j|\le4}\|S_{j'-1}u^h\|_{L^2_T(L^\infty)}\|\Delta_j\nabla_h\psi\|_{L^2_T(L^2)}\\
&\lesssim c_j2^{-j(s+1)}\|u^h\|_{L^2_T(L^\infty)}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}
\end{aligned}$$

and

$$\begin{aligned}
\|\Delta_j(T_{\nabla_h\psi}u^h)\|_{L^1_T(L^2)}&\lesssim\sum_{|j'-j|\le4}\|S_{j'-1}\nabla_h\psi\|_{L^2_T(L^\infty)}\|\Delta_ju^h\|_{L^2_T(L^2)}\\
&\lesssim c_j2^{-j(s+1)}\|\nabla_h\psi\|_{L^2_T(L^\infty)}\|\nabla u^h\|_{L^2_T(\dot H^s)}.
\end{aligned}$$

For $s>-\frac52$, Lemma 2.7 implies

$$\begin{aligned}
\|\Delta_j(R(u^h,\nabla_h\psi))\|_{L^1_T(L^2)}&\lesssim2^{\frac{3j}{2}}\sum_{j'\ge j-N_0}\|\Delta_{j'}u^h\|_{L^2_T(L^2)}\|\widetilde\Delta_{j'}\nabla_h\psi\|_{L^2_T(L^2)}\\
&\lesssim c_j2^{-j(s+1)}\|u^h\|_{L^2_T(\dot H^{3/2})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}.
\end{aligned}$$

As a consequence, we obtain

$$
\begin{aligned}
&\int_0^T \big|\big(\partial\Delta_j(u^h\cdot\nabla_h\psi)\ \big|\ \partial\Delta_j\psi\big)\big|dt\\
&\lesssim c_j^2 2^{-2js}\big(\|u^h\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\\
&\qquad\qquad + \|\nabla u\|_{L^2_T(\dot H^s)}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\dot B^{3/2}_{2,1})}\big)\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot H^s)}.
\end{aligned}
\tag{B.3}
$$

Let us turn to the estimate of $(\partial\Delta_j(u^3\partial_3\psi)\ |\ \partial\Delta_j\psi)$. By a standard commutator process (see [1] for instance), one writes

$$
\begin{aligned}
&(\partial\Delta_j(T_{u^3}\partial_3\psi)\ |\ \partial\Delta_j\psi)\\
&\quad=\sum_{|j'-j|\le 4}\big\{\big([\partial\Delta_j;S_{j'-1}u^3]\partial_3\Delta_{j'}\psi\ |\ \partial\Delta_j\psi\big)\\
&\qquad\qquad + \big([S_{j'-1}u^3-S_{j-1}u^3]\partial\Delta_j\partial_3\Delta_{j'}\psi)\ \big|\ \partial\Delta_j\psi\big)\big\}\\
&\quad + (S_{j-1}u^3\partial_3\Delta_j\partial\psi\ |\ \partial\Delta_j\psi).
\end{aligned}
\tag{B.4}
$$

The classical commutator estimates (see [1] for instance) imply that

$$
\begin{aligned}
&\sum_{|j'-j|\le 4}\int_0^T\big|\big([\partial\Delta_j;S_{j'-1}u^3]\partial_3\Delta_{j'}\psi\ \big|\ \partial\Delta_j\psi\big)\big|dt\\
&\lesssim\sum_{|j'-j|\le 4}\|S_{j'-1}\nabla u^3\|_{L^1_T(L^\infty)}\|\partial_3\Delta_{j'}\psi\|_{L^\infty_T(L^2)}\|\Delta_j\partial\psi\|_{L^\infty_T(L^2)}\\
&\lesssim c_j^2 2^{-2js}\|\nabla u^3\|_{L^1_T(L^\infty)}\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^s)}.
\end{aligned}
\tag{B.5}
$$

Since

$$
(S_{j-1}u^3\partial_3\Delta_j\partial\psi\ |\ \partial\Delta_j\psi) = -(S_{j-1}\partial_3 u^3\Delta_j\partial\psi\ |\ \partial\Delta_j\psi),
$$

the estimate (B.5) also holds for $\int_0^T|(S_{j-1}u^3\partial_3\Delta_j\partial\psi\ |\ \partial\Delta_j\psi)|dt$. We apply Lemma 2.7 to obtain

$$
\begin{aligned}
&\sum_{|j'-j|\le 4}\int_0^T\big|\big([S_{j'-1}u^3-S_{j-1}u^3]\partial\Delta_j\partial_3\Delta_{j'}\psi)\ |\ \partial\Delta_j\psi\big)\big|dt\\
&\lesssim\sum_{|j'-j|\le 4}\|S_{j'-1}\nabla u^3-S_{j-1}\nabla u^3\|_{L^1_T(L^\infty)}\|\partial\Delta_j\psi\|^2_{L^\infty_T(L^2)}\\
&\lesssim c_j^2 2^{-2js}\|\nabla u^3\|_{L^1_T(L^\infty)}\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^s)}.
\end{aligned}
$$

The above together with (B.4) leads to

$$
\int_0^T|(\partial\Delta_j(T_{u^3}\partial_3\psi)\,|\,\partial\Delta_j\psi)|dt \quad\lesssim\quad c_j^2 2^{-2js}\|\nabla u^3\|_{L^1_T(L^\infty)}\|\nabla\psi\|^2_{\widetilde{L}^\infty_T(\dot H^s)}.
\tag{B.6}
$$

Next, it follows from the proof of (2.7) and $\operatorname{div} u = 0$ that

$$
u^3 = 2^{-j}\big(\operatorname{div}_h\overrightarrow{\Delta}^h_j u^3 + \partial_3\Delta^3_j u^3\big) = 2^{-j}\big(\operatorname{div}_h\overrightarrow{\Delta}^h_j u^3 - \operatorname{div}_h\Delta^3_j u^h\big)
\tag{B.7}
$$

with

$$\overrightarrow{\Delta}_k^h a \overset{\text{def}}{=} (\Delta_{j,1}^h a, \Delta_{j,2}^h a), \quad \Delta_{j,n}^h a \overset{\text{def}}{=} \mathcal{F}^{-1}(\widetilde{\varphi}_n(2^{-j}\xi)\widehat{a}), \quad \Delta_j^3 a \overset{\text{def}}{=} \mathcal{F}^{-1}(\widetilde{\varphi}_3(2^{-j}\xi)\widehat{a}),$$

where $\widetilde{\varphi}_n(\xi) = -i\xi_n\widetilde{\varphi}(\xi)/|\xi|^2$ and $\widetilde{\varphi}$ is defined as in (2.7). Applying (B.7) and using integration by parts, we have

$$\begin{aligned}
&\big(\partial\Delta_j(T_{\partial_3\psi}u^3) \;\big|\; \partial\Delta_j\psi\big)\\
&\quad= \sum_{|j'-j|\le 4} \big(\Delta_j\partial(S_{j'-1}\partial_3\psi\Delta_{j'}u^3) \;\big|\; \Delta_j\partial\psi\big)\\
&\quad= -\sum_{|j'-j|\le 4} 2^{-j'}\big\{\big(\Delta_j\partial\big(S_{j'-1}\partial_3\nabla_h\psi\big(\overrightarrow{\Delta}_{j'}^h\Delta_{j'}u^3 - \Delta_{j'}^3\Delta_{j'}u^h\big)\big) \;\big|\; \Delta_j\partial\psi\big)\\
&\qquad\qquad + \big(\Delta_j\partial(S_{j'-1}\partial_3\psi\big(\overrightarrow{\Delta}_{j'}^h\Delta_{j'}u^3 - \Delta_{j'}^3\Delta_{j'}u^h\big)\big) \;\big|\; \Delta_j\partial\nabla_h\psi\big)\big\},
\end{aligned}$$

from which we deduce

$$\begin{aligned}
&\int_0^T \big|\big(\partial\Delta_j(T_{\partial_3\psi}u^3) \;\big|\; \partial\Delta_j\psi\big)\big|dt\\
&\quad\lesssim \sum_{|j'-j|\le 4} \|\Delta_{j'}u\|_{L^2_T(L^2)}\big(\|S_{j'-1}\partial_3\nabla_h\psi\|_{L^2_T(L^\infty)}\|\Delta_j\partial\psi\|_{L^\infty_T(L^2)}\\
&\qquad\qquad + \|S_{j'-1}\partial_3\psi\|_{L^\infty_T(L^\infty)}\|\Delta_j\partial\nabla_h\psi\|_{L^2_T(L^2)}\big)\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\|\nabla_h\psi\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})}\\
&\qquad\qquad + \|\psi\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}\|\nabla_h\psi\|_{L^2_T(\dot{H}^{s+1})}\big)\|\nabla u\|_{L^2_T(\dot{H}^s)}.
\end{aligned} \tag{B.8}$$

Finally, we note that

$$(\partial\Delta_j(R(u^3, \partial_3\psi) \mid \partial\Delta_j\psi) = \sum_{j'\ge j-N_0} \big(\Delta_j\partial(\Delta_{j'}u^3\widetilde{\Delta}_{j'}\partial_3\psi) \;\big|\; \Delta_j\partial\psi\big).$$

Then, by a similar proof to that for (B.8), we have

$$\begin{aligned}
&\int_0^T \big|\big(\partial\Delta_j(R(u^3, \partial_3\psi)) \;\big|\; \partial\Delta_j\psi\big)\big|dt\\
&\quad\lesssim 2^{\frac{3j}{2}} \sum_{j'\ge j-N_0} \|\Delta_{j'}u\|_{L^2_T(L^2)}\big(\|\widetilde{\Delta}_{j'}\partial_3\nabla_h\psi\|_{L^2_T(L^2)}\|\Delta_j\partial\psi\|_{L^\infty_T(L^2)}\\
&\qquad\qquad + \|\widetilde{\Delta}_{j'}\partial_3\psi\|_{L^\infty_T(L^2)}\|\Delta_j\partial\nabla_h\psi\|_{L^2_T(L^2)}\big)\\
&\quad\lesssim c_j^2 2^{-2js}\|\nabla\psi\|_{\widetilde{L}^\infty_T(\dot{H}^s)}\|u\|_{L^2_T(\dot{H}^{\frac{3}{2}})}\|\nabla_h\psi\|_{L^2_T(\dot{H}^{1+s})} \quad \text{for } s > -\frac{3}{2}.
\end{aligned} \tag{B.9}$$

The conclusion of the lemma follows from (B.3), (B.6), (B.8), and (B.9). □

PROOF OF LEMMA 4.7. Thanks to Bony's decomposition (2.8), we have

$$u\cdot\nabla\psi = T_u\nabla\psi + \mathcal{R}(u, \nabla\psi).$$

Similar to (B.4), we have

$$
\begin{aligned}
&(\Delta\Delta_j(T_u\nabla\psi) \mid \Delta\Delta_j\psi)\\
&\quad=\sum_{|j'-j|\le 4}\Big\{([\Delta\Delta_j;S_{j'-1}u]\cdot\nabla\Delta_{j'}\psi \mid \Delta\Delta_j\psi)\\
&\qquad\qquad+\big((S_{j'-1}u-S_{j-1}u)\cdot\nabla\Delta\Delta_j\Delta_{j'}\psi \;\big|\; \Delta\Delta_j\psi\big)\Big\}\\
&\qquad+(S_{j-1}u\cdot\nabla\Delta\Delta_j\psi \mid \Delta\Delta_j\psi).
\end{aligned}
\tag{B.10}
$$

Again by the time integrability of $\nabla_h\psi$ and $\partial_3\psi$, we deduce that

$$
\begin{aligned}
&\sum_{|j'-j|\le 4}\int_0^T |([\Delta\Delta_j;S_{j'-1}u]\cdot\nabla\Delta_{j'}\psi \mid \Delta\Delta_j\psi)|dt\\
&\quad\lesssim 2^j\sum_{|j'-j|\le 4}\big(\|S_{j'-1}\nabla u^h\|_{L^2_T(L^\infty)}\|\Delta_{j'}\nabla_h\psi\|_{L^2_T(L^2)}\\
&\qquad\qquad+\|S_{j'-1}\nabla u^3\|_{L^1_T(L^\infty)}\|\Delta_{j'}\partial_3\psi\|_{L^\infty_T(L^2)}\big)\|\Delta\Delta_j\psi\|_{L^\infty_T(L^2)}\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla u^h\|_{L^2_T(L^\infty)}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\\
&\qquad\qquad+\|\nabla u^3\|_{L^1_T(L^\infty)}\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\big)\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}.
\end{aligned}
$$

Next, from Lemma 2.7 one gets

$$
\begin{aligned}
&\sum_{|j'-j|\le 4}\int_0^T |((S_{j'-1}u-S_{j-1}u)\cdot\nabla\Delta\Delta_j\Delta_{j'}\psi \mid \Delta\Delta_j\psi)|dt\\
&\quad\lesssim 2^{-j}\sum_{|j'-j|\le 4}\big(\|S_{j'-1}\nabla u^h-S_{j-1}\nabla u^h\|_{L^2_T(L^\infty)}\|\Delta\Delta_{j'}\nabla_h\psi\|_{L^2_T(L^2)}\\
&\qquad\qquad+\|S_{j'-1}\nabla u^3-S_{j-1}\nabla u^3\|_{L^1_T(L^\infty)}\|\Delta\Delta_{j'}\partial_3\psi\|_{L^\infty_T(L^2)}\big)\\
&\qquad\qquad\times\|\Delta\Delta_j\psi\|_{L^\infty_T(L^2)}\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla u^h\|_{L^2_T(L^\infty)}\|\nabla_h\psi\|_{\widetilde L^2_T(\dot H^{1+s})}\\
&\qquad\qquad+\|\nabla u^3\|_{L^1_T(L^\infty)}\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\big)\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}.
\end{aligned}
$$

This along with (B.10) and $\operatorname{div} u=0$ leads to

$$
\begin{aligned}
&\int_0^T |(\Delta\Delta_j(T_u\nabla\psi) \mid \Delta\Delta_j\psi)|dt\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla u^h\|_{L^2_T(L^\infty)}\|\nabla_h\psi\|_{\widetilde L^2_T(\dot H^{1+s})}\\
&\qquad\qquad+\|\nabla u^3\|_{L^1_T(L^\infty)}\|\partial_3\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\big)\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}.
\end{aligned}
\tag{B.11}
$$

It is clear that

$$\begin{aligned}
&\|\Delta_j(\mathcal{R}(u^h,\nabla_h\psi))\|_{L^1_T(L^2)}\\
&\quad\lesssim \sum_{j'\geq j-N_0}\|\Delta_{j'}u^h\|_{L^2_T(L^2)}\|S_{j'+2}\nabla_h\psi\|_{L^2_T(L^\infty)}\\
&\quad\lesssim c_j2^{-j(2+s)}\|\nabla_h\psi\|_{L^2_T(L^\infty)}\|\nabla u^h\|_{L^2_T(\dot H^{1+s})}\quad\text{for } s>-2,
\end{aligned}$$

and hence

$$\begin{aligned}
\text{(B.12)}\quad &\int_0^T\big|\big(\Delta\Delta_j\big(\mathcal{R}(u^h,\nabla_h\psi)\big)\ \big|\ \Delta\Delta_j\psi\big)\big|dt\lesssim\\
&\qquad c_j^2 2^{-2js}\|\nabla_h\psi\|_{L^2_T(L^\infty)}\|\nabla u^h\|_{L^2_T(\dot H^{1+s})}\|\nabla\psi\|_{L^\infty_T(\dot H^{1+s})}\quad\text{for } s>-2.
\end{aligned}$$

Again by (B.7), and integration by parts, one obtains that

$$\begin{aligned}
&(\Delta\Delta_j(\mathcal{R}(u^3,\partial_3\psi))\ |\ \Delta\Delta_j\psi)\\
&\quad=\sum_{j'\geq j-N_0}\big(\Delta\Delta_j\big(\Delta_{j'}u^3\big)S_{j'+2}\partial_3\psi\ \big|\ \Delta\Delta_j\psi\big)\\
&\quad=-\sum_{j'\geq j-N_0}2^{-j'}\big\{\big(\Delta\Delta_j\big(\big(\overrightarrow{\Delta}^h_{j'}\Delta_{j'}u^3-\Delta^3_{j'}\Delta_{j'}u^h\big)S_{j'+2}\partial_3\nabla_h\psi\big)\ \big|\ \Delta\Delta_j\psi\big)\\
&\qquad\qquad+\big(\Delta\Delta_j\big(\overrightarrow{\Delta}^h_{j'}\Delta_{j'}u^3-\Delta^3_{j'}\Delta_{j'}u^h\big)S_{j'+2}\partial_3\psi\ \big|\ \Delta\Delta_j\nabla_h\psi\big)\big\}.
\end{aligned}$$

Then, by Lemma 2.7 one has

$$\begin{aligned}
&\int_0^T\big|\big(\Delta\Delta_j(\mathcal{R}(u^3,\partial_3\psi))\ \big|\ \Delta\Delta_j\psi\big)\big|dt\\
&\quad\lesssim 2^{2j}\sum_{j'\geq j-N_0}2^{-j'}\|\Delta_{j'}u\|_{L^2_T(L^2)}\big(\|S_{j'+2}\partial_3\nabla_h\psi\|_{L^2_T(L^\infty)}\|\Delta\Delta_j\psi\|_{L^\infty_T(L^2)}\\
&\qquad\qquad+\|S_{j'+2}\partial_3\psi\|_{L^\infty_T(L^\infty)}\|\Delta\Delta_j\nabla_h\psi\|_{L^2_T(L^2)}\big),
\end{aligned}$$

from which, one deduces that for $s>-2$

$$\begin{aligned}
&\int_0^T|(\Delta\Delta_j(\mathcal{R}(u^3,\partial_3\psi))\ |\ \Delta\Delta_j\psi)|dt\\
&\quad\lesssim 2^{j(2-s)}\Big(\sum_{j'\geq -N_0}2^{-j'(2+s)}\Big)\|\nabla_h\psi\|_{L^2_T(L^\infty)}\|\nabla u\|_{L^2_T(\dot H^{1+s})}\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\\
&\qquad+2^{j(3-s)}\Big(\sum_{j'\geq -N_0}2^{-j'(3+s)}\Big)\|\partial_3\psi\|_{L^\infty_T(L^\infty)}\|\nabla u\|_{L^2_T(\dot H^{1+s})}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\\
&\quad\lesssim c_j^2 2^{-2js}\big(\|\nabla_h\psi\|_{L^2_T(L^\infty)}\|\nabla\psi\|_{\widetilde L^\infty_T(\dot H^{1+s})}\\
&\qquad\qquad+\|\partial_3\psi\|_{L^\infty_T(L^\infty)}\|\nabla_h\psi\|_{L^2_T(\dot H^{1+s})}\big)\|\nabla u\|_{L^2_T(\dot H^{1+s})}.
\end{aligned}$$

Lemma 4.7 follows from the above together with (B.11) and (B.12). □

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