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An extended magnetostatic Born-Infeld model with a concave lower order term

Jun Chen1,a) and Xing-Bin Pan2,b)

1College of Computer and Information Engineering, Fujian Agriculture and Forestry University, Fuzhou 350002, Fujian, People’s Republic of China
2Department of Mathematics, East China Normal University, and NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, Shanghai 200062, People’s Republic of China

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This paper concerns an extended Born-Infeld model with a concave lower order term for the magnetostatic case. Three types of boundary value problems are considered: the boundary condition prescribing the tangential component of $A$, the natural boundary condition, and the boundary condition prescribing the tangential component of $\text{curl } A$. In each case we obtain existence and regularity of solutions for small boundary data. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4826995]

I. INTRODUCTION

A. Motivations

This paper concerns an extended Born-Infeld model with a concave lower order term. The motivations of our study come from the interests both in physics and in mathematics. The physical motivation comes from the Born-Infeld theory. In the magnetostatic case the action density in this theory is

$$\mathcal{L} = S(|\text{curl } A|^2),$$

where

$$S(t) = b^2 \left( \sqrt{1 + \frac{1}{b^2} t} - 1 \right), \quad t \geq 0,$$

with the scaling parameter $b > 0$. For the equation in the entire space $\mathbb{R}^3$, Yang has shown that the solutions with finite energy are trivial, namely the curl of solutions must be zero vector field. Therefore, extended Born-Infeld functionals which allow non-trivial solutions are necessary. In Ref. 12 we followed the ideas of Born-Infeld, Yang, Lin and Yang, Benci and Fortunato, and considered an extended model by introducing a convex lower order term into the magnetostatic Born-Infeld functional. It has been noticed in Ref. 12 that the extended functionals with either a convex or concave lower order term will have quite different behavior, and require different approaches. In this paper we study the functionals with concave lower order terms such as

$$\mathcal{S}^- [A] = \int_{\Omega} \left( S(|\text{curl } A|^2) - F(x, A) \right) \, dx$$

and

$$\mathcal{H}^- [A] = \int_{\Omega} \left( S(|\text{curl } A|^2) - F(x, A) \right) \, dx + 2 \int_{\partial \Omega} (\mathbf{D}^0 \times \mathbf{A}_T) \cdot \nu \, dS.$$

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a)E-mail: jchen_1001@163.com
b)E-mail: xbpan@math.ecnu.edu.cn

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Such type functionals are not convex, and in general they are unbounded from below, so the direct method used in Ref. 12 for convex functionals does not work in this case. We shall look for solutions of nonlinear eigenvalue problems associated with these functionals.

The mathematical motivation of this project mainly comes from our interest in the variational problems of functionals involving the operator curl. The difficulties in the study of the functionals of the form of (1.2) and (1.3) include lack of control on divergence and degree one growth of the leading order term for large $|\text{curl}\, \mathbf{A}|$. The effect of lower order terms and boundary conditions plays an important role in the existence and regularity theory, which needs to be studied systematically. Here we would like to mention some recent works on quasilinear systems involving the operator curl, see for instance Ref. 8, 15, and 17.

In this paper, we consider three types of boundary conditions.

(i) The Dirichlet problem: Find a critical point of the functional $\mathcal{S}^{-}$ with $F(x, \mathbf{A})$ replaced by $\beta F(x, \mathbf{A})$ for some positive constant $\beta$, under the Dirichlet boundary condition

$$\mathbf{A}_T = \mathbf{A}_T^0 \quad \text{on } \partial \Omega.$$  

More precisely, we look for a positive constant $\beta$ and a vector field $\mathbf{A}$ which solves the equation

$$\text{curl} \left( \frac{\beta}{2} \nabla_x F(x, \mathbf{A}) \right) = 0 \quad \text{in } \Omega$$

and satisfies the boundary condition (1.4).

(ii) The Neumann problem: Find a critical point $\mathbf{A}$ of the functional $\mathcal{N}^{-}$ with $F(x, \mathbf{A})$ and $\mathbf{D}_T^0$ replaced by $\beta F(x, \mathbf{A})$ and $\beta \mathbf{D}_T^0$ for some positive constant $\beta$, and without prescribing boundary data. More precisely we look for a positive constant $\beta$ and a vector field $\mathbf{A}$ which is a solution of (1.5) subjected to the following natural boundary condition

$$\mathbf{S}'(\mathbf{A}) \cdot (\text{curl} \mathbf{A}) = \beta \mathbf{D}_T^0 \quad \text{on } \partial \Omega.$$  

(iii) The third type problem: Find solutions of the extended Born-Infeld models (see (1.14) and (1.15)) with prescribing tangential component of curl $\mathbf{A}$, i.e.:

$$\text{curl} \mathbf{A}_T = \mathbf{B}_T^0 \quad \text{on } \partial \Omega.$$  

B. Assumptions and main results

The majority of our method is valid for a general convex function $F(x, \mathbf{A})$, here we shall represent our main results in the special case where $F(x, \mathbf{A}) = a(x)|\mathbf{A}|^2$. Let $0 < \alpha < 1$. We shall make the following assumptions:

$(A1)$ $\Omega$ is a simply-connected bounded domain in $\mathbb{R}^3$ without holes, and with a $C^4$ boundary.

$(A2)$ $a(x) \in C^{1,1}(\Omega)$, $a_0 = \min_{x \in \Omega} a(x) > 0$.

$(A3)$ $\mathbf{A}_T^0 \in TC^{2-\alpha}(\partial \Omega, \mathbb{R}^3)$, and $\nu \cdot \text{curl} \mathbf{A}_T^0 = 0$ on $\partial \Omega$.

$(A4)$ $\mathbf{D}_T^0 \in TC^{1-\alpha}(\partial \Omega, \mathbb{R}^3)$, and $\nu \cdot \text{curl} \mathbf{D}_T^0 \in C^{1-\alpha}(\partial \Omega)$.

$(A5)$ $\mathbf{B}_T^0 \in TC^{2-\alpha}(\partial \Omega, \mathbb{R}^3)$.

To get the higher regularity of solutions we need a stronger condition on $a(x)$:

$(A2')$ $a(x) \in C^{2-\alpha}(\tilde{\Omega})$, $a_0 = \min_{x \in \tilde{\Omega}} a(x) > 0$.

For the nonlinear eigenvalue problem with the Dirichlet boundary condition

$$\begin{cases}
\text{curl} \left( \mathcal{S}'(\mathbf{A}) \text{curl} \mathbf{A} \right) - \beta a(x) \mathbf{A} = 0 & \text{in } \Omega, \\
\mathbf{A}_T = \mathbf{A}_T^0 & \text{on } \partial \Omega,
\end{cases}$$

(1.8)
we have the following:

**Theorem 1.1.** Assume that $\Omega$ and $a(x)$ satisfy (A1) and (A2), respectively, with $0 < \alpha < 1$. Then there exists $R_1 > 0$ such that for any $A_0^0$ satisfying (A3) with

$$\|A_0^0\|_{C^{1+\alpha}(\Omega)} \leq R_1,$$

there exists a positive constant $\beta$ such that (1.8) has a solution $A$, and

$$A = v + \nabla \phi \in C^{2,\alpha}_0(\bar{\Omega}, \text{div}0, A_0^0) + \text{grad} C^{2,\alpha}_0(\bar{\Omega}).$$

(1.10)

In Theorem 1.1, if we replace the condition (A2) by (A2'), then the solution $A \in C^{2,\alpha}(\bar{\Omega}, \mathbb{R}^3)$; see Corollary 3.10.

For the nonlinear eigenvalue problem with the natural boundary condition

$$\begin{aligned}
\text{curl} \left( S(|\text{curl} A|^2) \text{curl} A \right) - \beta a(x)A &= 0 \quad \text{in } \Omega, \\
S(|\text{curl} A|^2) (\text{curl} A)_T &= \beta D_T^0 \quad \text{on } \partial \Omega,
\end{aligned}$$

(1.11)

we have the following:

**Theorem 1.2.** Assume that $\Omega$ and $a(x)$ satisfy (A1) and (A2), respectively, with $0 < \alpha < 1$. Then there exists $R_2 > 0$ such that for any $D_T^0$ satisfying (A4) with

$$\|D_T^0\|_{C^{1+\alpha}(\Omega)} \leq R_2,$$

(1.12)

there exists a positive constant $\beta$ such that (1.11) has a solution $A$, and

$$A = v + \nabla \phi \in C^{2,\alpha}_0(\bar{\Omega}, \text{div}0) + \text{grad} C^{2,\alpha}(\bar{\Omega}).$$

(1.13)

For the third type problem we consider both the system

$$\begin{aligned}
\text{curl} \left( S(|\text{curl} A|^2) \text{curl} A \right) + a(x)A &= 0 \quad \text{in } \Omega, \\
(\text{curl} A)_T &= B_T^0 \quad \text{on } \partial \Omega,
\end{aligned}$$

(1.14)

and the system

$$\begin{aligned}
\text{curl} \left( S(|\text{curl} A|^2) \text{curl} A \right) - a(x)A &= 0 \quad \text{in } \Omega, \\
(\text{curl} A)_T &= B_T^0 \quad \text{on } \partial \Omega.
\end{aligned}$$

(1.15)

The functional associated with (1.14) has a convex lower order term, and that associated with (1.15) has a concave lower order term. Existence of critical points will be proved using the implicit function theorem, and it will be shown that if the boundary datum $B_T^0$ is small then (1.14) has always a solution, while (1.15) has a solution provided $\lambda = 2$ is not an eigenvalue of the following problem:

$$\begin{aligned}
\text{curl} \left( \frac{1}{a(x)} \text{curl} u \right) - \lambda u &= 0 \quad \text{in } \Omega, \\
\text{div} u &= 0 \quad \text{in } \Omega, \\
u_T &= 0 \quad \text{on } \partial \Omega.
\end{aligned}$$

(1.16)

**Theorem 1.3.** In addition to the conditions (A1) and (A2'), assume furthermore $\partial \Omega$ is of class $C^{5,\alpha}$ with $0 < \alpha < 1$. Then we have the following conclusions.

(i) There exists $R_3 > 0$ such that for any $B_T^0$ satisfying (A5) with

$$\|B_T^0\|_{C^{1+\alpha}(\partial \Omega)} \leq R_3,$$

then (1.14) has a solution $A$, and

$$A = v + \nabla \phi \in C^{5,\alpha}_0(\bar{\Omega}, \text{div}0) + \text{grad} C^{2,\alpha}(\bar{\Omega}).$$

(1.17)

(ii) The same conclusion is true for problem (1.15) if $\lambda = 2$ is not an eigenvalue of (1.16).
C. Outlines of our approach

We start with the general functionals of the form (1.2) and (1.3), as most part of our approach is valid for them. The spaces mentioned in this subsection will be given in Subsection II A.

1. The Dirichlet problem

As in Ref. 12 we modify the function $S(t)$ for $t > K$, with $0 < K < b^2$, to get a strictly increasing function $S_K(t)$ which has a linear growth in $t$ at infinity, and we consider the corresponding modified functional

$$S^-_K[A] = \int_{\Omega} (S_K(|\text{curl} A|^2) - F(x, A)) \, dx,$$

and look for critical points of $S^-_K$ in $H_0^{2, p}(\Omega, \text{curl}, A^0_\Omega)$. They turn out to be the critical points of the original functional $S^-$ if

$$\|\text{curl} A\|_{L^\infty(\Omega)} \leq \sqrt{K}. \quad (1.19)$$

The functional $S^-_K$ is neither convex nor coercive, and it is lack of compactness on the “natural admissible set” $H_0^{2, p}(\Omega, \text{curl}, A^0_\Omega)$. To resume the compactness one may consider the space $H_0^{2, p}(\Omega, \text{curl}, \text{div} 0, A^0_\Omega)$. Then a critical point $A$, if exists, satisfies the following integral equation

$$\int_{\Omega} (2S'_{K}(|\text{curl} A|^2) \text{curl} A \cdot \text{curl} w - \nabla_x F(x, A) \cdot w) \, dx = 0, \quad \forall w \in C^1_0(\bar{\Omega}, \text{div} 0),$$

from which we cannot derive that $A$ is a weak solution of the Euler-Lagrange equation of $S^-_K$ since the set $C^1_0(\bar{\Omega}, \text{div} 0)$ is not dense in $H_0^{2, p}(\Omega, \text{curl}, A^0_\Omega)$.

To overcome this difficulty we shall adapt the method developed by Benci and Fortunato, and then we can find a solution of (1.5) for some positive constant $\beta$, with $A$ having the following decomposition:

$$A = v + \nabla \phi, \quad (1.20)$$

where $v$ is a divergence-free vector field.

Step 1. Define functionals

$$F[A] = \int_{\Omega} F(x, A) \, dx, \quad \mathcal{F}[v, \phi] = F[v + \nabla \phi]. \quad (1.21)$$

For any fixed $v \in H_0^{2, p}(\Omega, \text{curl}, \text{div} 0, A^0_\Omega)$, the functional $\mathcal{F}[v, \cdot]$ has a unique minimizer $\phi_v$ on $W^{1, p}_0(\Omega)$, and hence $\phi_v$ is a weak solution of

$$\begin{cases}
\text{div}(\nabla_x F(x, v + \nabla \phi_v)) = 0 & \text{in } \Omega, \\
\phi_v = 0 & \text{on } \partial \Omega.
\end{cases} \quad (1.22)$$

Step 2. If the functional

$$\mathcal{I}[v] = S^-_K[v + \nabla \phi_v] = \int_{\Omega} S_K(|\text{curl} v|^2) \, dx - \mathcal{F}[v, \phi_v],$$

has a minimizer $v$ in $H_0^{2, p}(\Omega, \text{curl}, \text{div} 0, A^0_\Omega)$, then using (1.22) we can show that $A = v + \nabla \phi_v$ is a weak solution of the Euler-Lagrange equation for $S^-_K[A]$. Unfortunately, the functional $\mathcal{I}$ may not be bounded from below. Following the idea of Benci and Fortunato, we introduce a bounded function $G$ and consider the truncated functional

$$S^-_{K,G}[v] = \int_{\Omega} S_K(|\text{curl} v|^2) \, dx - G(\mathcal{F}[v, \phi_v]), \quad (1.23)$$
where \( G \) satisfies

\[(G) \quad G : \mathbb{R} \to \mathbb{R} \text{ is a } C^1 \text{ function, } G(t) > 0 \text{ for all } t, \text{ and} \]

\[M_1 = \sup_{t \in \mathbb{R}} G(t) < +\infty, \quad M_2 = \sup_{t \in \mathbb{R}} G'(t) < +\infty.\]

Such functions \( G \) always exist. For example, we can choose \( G(t) = \arctan t \). Note that multiplying \( G \) by a small positive constant we can require \( M_2 \) to be sufficiently small, and this fact will be used in (3.19) and (3.28).

Now the functional \( S_{K, G} \) is bounded from below, and we can show that \( S_{K, G} \) has a minimizer \( v_K \in \mathcal{S}_{2-p}^1(\Omega, \text{curl}, \text{div}0, A_0^0) \). If \( \Omega \) has no holes, using (1.22) we can show that there exists a constant \( \beta \) such that \( A_K = v_K + \nabla \phi_{v_K} \) is a weak solution of the system

\[
\begin{align*}
\text{curl} \left( S'_{K} \left( \| \text{curl} A \|^2 \right) \text{curl} A \right) - \frac{\beta}{2} \nabla_2 F(x, A) &= 0 \quad \text{in } \Omega, \\
A_T &= A_T^0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.24)

For the definition of weak solutions of (1.24) we refer to Ref. 12, Definition 2.3.

The last step in our approach is the verification of (1.19). This will be done in the special case where \( F(x, A) = a(x) \| A \|^2 \).

2. The Neumann problem

Step 1. Define the functional

\[F^\gamma[v, \phi] = F[v, \phi] - 2 \int_{\partial \Omega} (D_T^0 \times (\nabla \phi)_T) \cdot \nu \, dS. \quad (1.25)\]

For any fixed \( v \in \mathcal{S}_{2-p}^1(\Omega, \text{curl}, \text{div}0, g) \), the functional \( F^\gamma[v, \cdot] \) has a unique minimizer \( \phi^\gamma_v \) on \( W^{1,p}(\Omega) \). Hence \( \phi^\gamma_v \) is a weak solution of

\[
\begin{align*}
\text{div}(\nabla_2 F(x, v + \nabla \phi)) &= 0 \quad \text{in } \Omega, \\
v \cdot \nabla_2 F(x, v + \nabla \phi) &= 2v \cdot \text{curl} D_T^0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.26)

Step 2. Define the functional

\[I^\gamma[v, \phi] = H_{K}^\gamma[v + \nabla \phi^\gamma_v] = \int_{\Omega} S_K(\| \text{curl} v \|^2) \, dx - F^\gamma[v, \phi^\gamma_v] + 2 \int_{\partial \Omega} (D_T^0 \times v_T) \cdot \nu \, dS, \]

may also be unbounded from below, so we consider the truncated functional

\[H_{K, G}[v] = \int_{\Omega} S_K(\| \text{curl} v \|^2) \, dx - (F^\gamma[v, \phi^\gamma_v] - 2 \int_{\partial \Omega} (D_T^0 \times v_T) \cdot \nu \, dS), \quad (1.27)\]

where \( G \) satisfies the condition \((G)\). Because of noncoercivity of \( H_{K, G} \) in a natural admissible space \( \mathcal{S}_{2-p}(\Omega, \text{curl}, \text{div}0) \cap \mathcal{H}^1_1(\Omega) \), we consider the minimization problem of \( H_{K, G} \) in the subspace of \( \mathcal{S}_{2-p}(\Omega, \text{curl}, \text{div}0) \cap \mathcal{H}^1_1(\Omega) \) with an additional boundary condition

\[v \cdot \nu = g \quad \text{on } \partial \Omega. \quad (1.28)\]

We shall prove that \( H_{K, G} \) has a minimizer \( v_K \). Moreover, if \( \Omega \) is simply connected, using (1.26) we can show that there exists a constant \( \beta \) such that \( A_K = v_K + \nabla \phi_{v_K} \) is a weak solution of the system

\[
\begin{align*}
\text{curl} \left( S'_{K} \left( \| \text{curl} A \|^2 \right) \text{curl} A \right) - \frac{\beta}{2} \nabla_2 F(x, A) &= 0 \quad \text{in } \Omega, \\
S'_{K} \left( \| \text{curl} A \|^2 \right) (\text{curl} A)_T &= \beta D_T^0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.29)

For the definition of weak solutions of (1.29) we refer to Ref. 12, Definition 2.4.

The verification of (1.19) will be also done in the special case where \( F(x, A) = a(x) \| A \|^2 \).

This paper is organized as follows. Section II contains preliminary results that will be used, and a list of conditions on \( F \). The Dirichlet problem is studied in Sec. III, where we first prove
the existence of critical points of the truncated functional $S_{K,G}^-\cdot$ in Subsection III A, and give the $L^2$ and $C^{2,\alpha}$ estimates in Subsection III B, then we obtain the solutions of (1.8) and prove Theorem 1.1 in Subsection III C. The Neumann problem is treated in Sec. IV. We prove the existence and regularity of the critical points of the truncated functional $H_{K,G}$ in Subsections IV A and IV B, and obtain the solutions of (1.11) and prove Theorem 1.2 in Subsection IV C. The connection between Dirichlet problem and Neumann problem is shown in Subsection IV D. The third type problem and the corresponding gauge invariant problem is dealt with in Sec. V using the implicit function theorem.

II. PRELIMINARIES

A. Notation

We shall adapt the notation used in Ref. 12. As in Ref. 12, $\Omega$ is a bounded domain in $\mathbb{R}^3$, and $\nu$ denotes the unit outer normal vector field of the domain boundary $\partial\Omega$. $A_T$ denotes the tangential component of a vector field $A$ on $\partial\Omega$:

$$A_T = A - (A \cdot \nu)\nu.$$

For $1 < p < +\infty$, $p'$ denotes the conjugate index of $p$, i.e., $1/p + 1/p' = 1$. We use $C^{k,\alpha}(\Omega)$, $W^{k,p}(\Omega)$, $H^s(\Omega)$, etc., to denote the usual Hölder spaces and Sobolev spaces, and use $C^{k,\alpha}(\bar{\Omega}, \mathbb{R}^3)$, $W^{k,p}(\Omega, \mathbb{R}^3)$, $H^s(\Omega, \mathbb{R}^3)$, etc., to denote the corresponding Hölder spaces and Sobolev spaces for vector fields. Denote

$$H_1(\Omega) = \{v \in L^2(\Omega, \mathbb{R}^3) : \text{curl} v = 0, \ div v = 0 \text{ in } \Omega, v \cdot \nu = 0 \text{ on } \partial\Omega\},$$

$$H_2(\Omega) = \{v \in L^2(\Omega, \mathbb{R}^3) : \text{curl} v = 0, \ div v = 0 \text{ in } \Omega, v_T = 0 \text{ on } \partial\Omega\},$$

and denote by $H^j_\perp(\Omega), j = 1, 2$, the orthogonal complement of $H_j(\Omega)$ in $L^2(\Omega, \mathbb{R}^3)$ with respect to the $L^2$ inner product. The following spaces have been introduced in Ref. 12 with $1 \leq p, q < +\infty$:

$$\mathcal{S}^p(\Omega, \text{curl}) = \{u \in L^p(\Omega, \mathbb{R}^3) : \text{curl} u \in L^q(\Omega, \mathbb{R}^3)\},$$

$$\mathcal{S}^p(\Omega, \text{div}) = \{u \in L^p(\Omega, \mathbb{R}^3) : \text{div} u \in L^q(\Omega)\},$$

$$\mathcal{S}^p(\Omega, \text{curl, div}) = \mathcal{S}^p(\Omega, \text{curl}) \cap \mathcal{S}^p(\Omega, \text{div}).$$

For simplicity we denote

$$\mathcal{G} = \mathcal{S}^2(\Omega, \text{curl, div}) \cap H^1_\perp(\Omega),$$

$$\mathcal{G}_0 = \mathcal{S}^2(\Omega, \text{curl, div}) \cap H^1_\perp(\Omega),$$

$$\mathcal{G}_0 = \mathcal{S}^2(\Omega, \text{curl, div}) \cap H^1_\perp(\Omega),$$

which will be used in Subsections III B and IV B, respectively.

If $X(\Omega, *)$ denotes a set of vector fields on $\Omega$, with the asterisk stands for a description on the set, then we use the notation $X(\Omega, *, \text{div})$, $X(\Omega, *, \text{curl})$, $X(\Omega, *, g)$, $X(\Omega, *, *)$, and $X(\Omega, *, *)$ as in Ref. 12. If $Y(\partial\Omega, *)$ is a set of vector fields on $\partial\Omega$, then

$$TY(\partial\Omega, *) = \{w \in Y(\partial\Omega, *) : \nu \cdot w = 0 \text{ on } \partial\Omega\}.$$

If $X(\Omega)$ denotes a space of functions defined on $\Omega$, then we write

$$X(\Omega) = \{\phi \in X(\Omega) : \int_\Omega \phi \, dx = 0\}.$$

The constants $c$, $c'$, $C$, and $C'$ denote generic constants which may vary from line to line.
B. On the spaces of vector fields

In this subsection we collect some useful facts about the spaces of vector fields. Part of the results are well-known, see, for instance Refs. 1, 9–11, 13–15, and 19.

Lemma 2.1. Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary, \( A_0^p \in TH^{1,2}(\partial \Omega, \mathbb{R}^3) \) and \( g \in H^{1,2}(\partial \Omega) \). Then for any \( 1 \leq p \leq 6 \) we have

\[
\begin{align*}
\mathcal{S}_1^{2,2}(\Omega, \text{curl}, \text{div}0, A_0^p) &= \mathcal{S}_1^{2,2}(\Omega, \text{curl}, \text{div}0, A_0^p), \\
\mathcal{S}_2^{2,2}(\Omega, \text{curl}, \text{div}0, g) &= \mathcal{S}_2^{2,2}(\Omega, \text{curl}, \text{div}0, g).
\end{align*}
\] (2.1)

Proof. The equalities (2.1) hold in the sense that the norms are equivalent. We give only the proof of the first equality. Assume \( \mathbf{v} \in \mathcal{S}_1^{2,2}(\Omega, \text{curl}, \text{div}0, A_0^p) \). When \( 2 \leq p \leq 6 \) then obviously \( \mathbf{v} \in \mathcal{S}_1^{2,2}(\Omega, \text{curl}, \text{div}0, A_0^p) \). When \( 1 \leq p < 2 \), we use the interpolation inequality

\[
\| \mathbf{u} \|_{L^p(\Omega)} \leq \varepsilon \| \mathbf{u} \|_{L^2(\Omega)} + C(\varepsilon) \| \mathbf{u} \|_{L^{p/2}(\Omega)}.
\] (2.2)

and div-curl-gradient-inequalities of vector fields (see, for instance, Ref. 13, p. 212; and Ref. 9) to find

\[
\| \mathbf{v} \|_{L^p(\Omega)} \leq C(\Omega) \left( \| \text{curl} \mathbf{v} \|_{L^2(\Omega)} + \| A_0^p \|_{H^{1/2}(\partial \Omega)} \right) + C'(\Omega) \| \mathbf{v} \|_{L^2(\Omega)}.
\]

Hence \( \mathbf{v} \in \mathcal{S}_1^{2,2}(\Omega, \text{curl}, \text{div}0, A_0^p) \).

Next assume \( \mathbf{v} \in \mathcal{S}_2^{2,2}(\Omega, \text{curl}, \text{div}0, A_0^p) \). Applying div-curl-gradient-inequalities of vector fields we find \( \mathbf{v} \in H^1(\Omega, \mathbb{R}^3) \), thus \( \mathbf{v} \in L^p(\Omega, \mathbb{R}^3) \) by the Sobolev embedding, and hence \( \mathbf{v} \in \mathcal{S}_1^{2,2}(\Omega, \text{curl}, \text{div}0, A_0^p) \).

We also see that the norms in the two spaces are equivalent. \( \square \)

From Lemma 2.1 we know that for \( 1 \leq p \leq 6 \), \( \mathcal{S}_0 \) and \( \mathcal{S}_3 \) are both well-defined as subspaces of \( L^2(\Omega, \mathbb{R}^3) \).

Recall the Poincaré inequality for functions

\[
\| \phi \|_{L^2(\Omega)} \leq C_p(\Omega) \| \nabla \phi \|_{L^2(\Omega)}, \quad \forall \phi \in H^1_0(\Omega). \] (2.3)

For vector fields we have a similar inequality which may also be called a variation of Poincaré inequality.

Lemma 2.2. Let \( 1 \leq p \leq 6 \). Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary. Then there exist positive constants \( C_{vp}(\Omega) \) and \( C(\Omega) \) such that if either \( \mathbf{u} \in \mathcal{S}_0 \), or \( \mathbf{u} \in \mathcal{S}_3 \), we have

\[
\| \mathbf{u} \|_{L^p(\Omega)} \leq C_{vp}(\Omega) \| \text{curl} \mathbf{u} \|_{L^2(\Omega)}, \] (2.4)

\[
\| \nabla \mathbf{u} \|_{L^p(\Omega)} \leq C(\Omega) \| \text{curl} \mathbf{u} \|_{L^2(\Omega)}. \] (2.5)

Proof. We only prove (2.4) for \( \mathbf{u} \in \mathcal{S}_0 \). Suppose it is not true, applying Lemma 2.1, then there exists a sequence \( \{ \mathbf{u}_n \} \subset \mathcal{S}_0 = \mathcal{S}_2^{2,2}(\Omega, \text{curl}, \text{div}0) \cap H^2(\Omega, \mathbb{R}^3) \), such that \( \| \mathbf{u}_n \|_{L^2(\Omega)} = 1 \), but \( \| \text{curl} \mathbf{u}_n \|_{L^2(\Omega)} \to 0 \). Hence \( \{ \mathbf{u}_n \} \) is bounded in \( H^1(\Omega, \mathbb{R}^3) \) by the div-curl-gradient inequalities of vector fields, and there exists a subsequence \( \{ \mathbf{u}_{n_j} \} \), such that as \( j \to \infty \), \( \mathbf{u}_{n_j} \to \mathbf{u}_0 \) weakly in \( H^1(\Omega, \mathbb{R}^3) \) and strongly in \( L^2(\Omega, \mathbb{R}^3) \). Thus we have

\[
\| \mathbf{u}_0 \|_{L^2(\Omega)} = 1,
\] (2.6)

and \( \mathbf{u}_0 \in H^2(\Omega) \). On the other hand, \( H^2(\Omega, \mathbb{R}^3) \) is weakly closed in \( \mathcal{S}_0^{2,2}(\Omega, \text{curl}, \text{div}0) \), thus \( \mathbf{u}_0 \in \mathcal{S}_0^{2,2}(\Omega, \text{curl}, \text{div}0) \). Hence \( \mathbf{u}_0 = \mathbf{0} \) a.e. in \( \Omega \), which contradicts with (2.6). Hence (2.4) is proved.

Applying the div-curl-gradient inequalities of vector fields to \( \mathbf{u} \in \mathcal{S}_0^{2,2}(\Omega, \text{curl}, \text{div}0) \) we have

\[
\| \nabla \mathbf{u} \|_{L^p(\Omega)} \leq C(\Omega) \left( \| \text{curl} \mathbf{u} \|_{L^2(\Omega)} + \| \mathbf{u} \|_{L^2(\Omega)} \right).
\]

Using this and (2.4) we get (2.5). \( \square \)
Lemma 2.3. Let $1 \leq p \leq 6$. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary. Then there exist positive constants $c = c(p, \Omega)$ and $C = C(p, \Omega)$ such that for any $u \in \dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)$ we have
\begin{equation}
    c\|u\|_{\dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)} \leq \|\nabla u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{\dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)},
\end{equation}
and for any $u \in \dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)$ we have
\begin{equation}
    c\|u\|_{\dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)} \leq \|\nabla u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{\dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)}.
\end{equation}
Hence $\|\nabla u\|_{L^2(\Omega)}$ is an equivalent norm in $\dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)$ and in $\dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)$.

Proof. We only prove (2.7). For any $u \in \dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)$, we apply the div-curl-gradient inequalities of vector fields. Noting that $\text{div}u = 0$ and $u \times \nu = 0$ on $\partial\Omega$, we get
\begin{equation}
    \|\nabla u\|_{L^2(\Omega)} \leq C_1(\Omega) \left(\|\text{curl} u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right).
\end{equation}
If $p \geq 2$ then the right inequality of (2.7) follows from (2.9). If $1 \leq p < 2$, then the right inequality of (2.7) follows from (2.9) and the interpolation inequality (2.2).

On the other hand we have a Poincaré type inequality: For $1 \leq p \leq 6$ and any $u \in \dot{S}^{2,p}_{00}(\Omega, \text{curl}, \text{div}0)$, we have
\begin{equation}
    \|u\|_{L^p(\Omega)} \leq C(p, \Omega)\|\nabla u\|_{L^2(\Omega)}.
\end{equation}
This inequality can be proved by contradiction, see for instance the proof of Lemma 2.2. So the left inequality is true. \hfill \Box

C. Conditions on $F$

We list the conditions on $F(x, z)$ that will be used later. Let $1 < p < +\infty$.

(F1) $F : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is strictly convex in $z$ and has continuous partial derivatives $\nabla^2_x F$ and $\nabla^2_z F$.

(F2) There exist positive constants $c_1, c_2, C_1, C_2$, such that for any $x \in \Omega$ and $z \in \mathbb{R}^3$,
\begin{align*}
    c_1|x|^p - c_2 & \leq F(x, z) \leq C_1(|z|^p + 1), \\
    |\nabla_x F(x, z)| & \leq C_2(|z|^{p-1} + 1).
\end{align*}

D. The modified functionals

Let $S(t)$ be the function defined in (1.1) and let $\Phi(t) = t(S'(t))^2$. Then $\Phi(t) \in C^\infty([0, +\infty))$ is a positive, strictly increasing and strictly concave function on $(0, +\infty)$. Let
\begin{equation}
    f(\rho) \equiv \frac{1}{S'(\Phi^{-1}(\rho))} = 2\left(1 - \frac{4}{b^2}\right)^{-1/2}.
\end{equation}

Lemma 2.4 (Ref. 12, Lemma 3.2). Given a constant $K$ with $0 < K < b^2$, we can find a small $\delta > 0$ and construct a positive function $S_K(t) \in C^3([0, +\infty))$ such that the following conclusions are true:

(i) $S_K(t) = S(t)$ for $t \in [0, K]$ and $S_K(t) = at^2 + b_1$ for $t \geq K + \delta$, where $a_K$ and $b_1$ are positive constants. Moreover, we have
\begin{equation}
    S_K(t) > 0, \quad S_K'(t) + 2tS_K''(t) > 0, \quad \forall t \geq 0.
\end{equation}

(ii) Let
\begin{equation}
    \Phi_K(t) = t\left(S_K'(t)\right)^2, \quad f_K(\rho) = \frac{1}{S_K'(\Phi_K^{-1}(\rho))}.
\end{equation}
Then \( \Phi_K \) is \( C^2 \) on \([0, +\infty)\) and it is a strictly increasing function. \( f_K \) is \( C^2 \) on \([0, +\infty)\), positive-valued, and \( f_K(\rho) = 1/a_K \) for all \( \rho \geq a_K^2(K + \delta) \). Moreover,

\[
2\beta'(t) - \frac{\Phi_K(t)}{t} \geq l(K) > 0, \quad \forall t \in [K, K + \delta],
\]

(2.12)

where \( l(K) \) is a constant depending on \( K \), and \( \lambda(K, \delta) \) is a constant depending on \( K \) and \( \delta \).

**Lemma 2.5.** Let \( f \) and \( f_K \) be the functions stated above. Then we have the following conclusions:

(i) \( u = S'(\|B\|^2)B \) if and only if \( B = f(\|u\|^2)u \).

(ii) \( u = S_K'(\|B\|^2)B \) if and only if \( B = f_K(\|u\|^2)u \).

**Proof.** (i) If \( u = S'(\|B\|^2)B \) then we have \( \|u\|^2 = \Phi(\|B\|^2) \), which is equivalent to

\[
|B|^2 = \Phi^{-1}(\|u\|^2) = \frac{4\|u\|^2}{1 - \frac{4}{b^2}\|u\|^2}.
\]

Hence we obtain

\[
B = \frac{u}{S'(\|B\|^2)} = \frac{u}{S'(\Phi^{-1}(\|u\|^2))} = f(\|u\|^2)u = 2(1 - \frac{4}{b^2}\|u\|^2)^{-1/2}u.
\]

If \( B = f(\|u\|^2)u \), we only need to calculate backward. (ii) Replacing \( S, \Phi \) and \( f \) by \( S_K, \Phi_K \) and \( f_K \) in the proof of (i), the conclusion is followed. \( \square \)

III. THE DIRICHLET PROBLEM

A. Existence of critical points of \( S_{K,G}^- \)

Let \( F[A] \) and \( F[v, \phi] \) be the functionals introduced in (1.21), and \( S_{K,G}^- \) be the truncated functional introduced in (1.23). We show that \( S_{K,G}^- \) has critical points.

1. Minimization of \( F \)

**Proposition 3.1.** Let \( 1 < p < +\infty \). Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary, and \( F \) satisfies (F1) and (F2). Then for any \( v \in S^1_p(\Omega, \text{curl}, \text{div}0, A_0^0) \), the functional \( F[v, \cdot] \) has a unique minimizer \( \phi_n \) in \( W_0^{1,p}(\Omega) \) with

\[
F[v, \phi_n] = a(F, v) = \inf_{\phi \in W_0^{1,p}(\Omega)} F[v, \phi].
\]

**Proof.** Let \( \{\phi_n\} \subset W_0^{1,p}(\Omega) \) be a minimizing sequence. By (F2), we have

\[
\int_\Omega (c_1|v + \nabla \phi_n|^p - c_2) \, dx \leq \int_\Omega F(x, v + \nabla \phi_n) \, dx \leq a(F, v) + o(1).
\]

Thus \( \{\nabla \phi_n\} \) is bounded in \( L^p(\Omega, \mathbb{R}^3) \), and hence \( \{\phi_n\} \) is bounded in \( W_0^{1,p}(\Omega) \). After passing to a subsequence we may assume that, \( \phi_n \to \phi \) weakly in \( W_0^{1,p}(\Omega) \). By the definition of \( a(F, v) \) we have \( F[v, \phi_n] \geq a(F, v) \).

The weakly lower semi-continuity of the functional \( F[v, \phi] \) is derived from the convexity of \( F \) by (F1). Therefore, we have

\[
F[v, \phi_n] \leq \liminf_{n \to \infty} F[v, \phi_n] = a(F, v).
\]

So \( \phi \) is a minimizer. The uniqueness follows from the strict convexity of \( F \). \( \square \)

Since \( \phi \) is the minimizer of the functional \( F[v, \phi] \) in \( W_0^{1,p}(\Omega) \),

\[
\int_\Omega \nabla \phi F(x, v + \nabla \phi_n) \cdot \nabla \psi \, dx = 0, \quad \forall \psi \in W_0^{1,p}(\Omega).
\]

(3.1)
Lemma 3.2. Let $1 < p < +\infty$. Assume that $F$ satisfies (F1) and (F2). Then the mapping $A \mapsto \nabla F(x, A(x))$ is a bounded mapping from $L^p(\Omega, \mathbb{R}^3)$ into $L^p(\Omega, \mathbb{R}^3)$, and the functional $\mathcal{F}[A]$ is continuous in $L^p(\Omega, \mathbb{R}^3)$.

Lemma 3.3. Let $1 < p < 6$. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, and $A^0 \in TH^{1/2}(\partial \Omega, \mathbb{R}^3)$. Assume that $F$ satisfies (F1) and (F2). Then the functional $v \mapsto \mathcal{F}[v, \phi_\ast]$ is weakly continuous on $\delta^2_{1/p}(\Omega, \text{curl}, \text{div}0, A^0)$.

Proof. Since $A^0 \in TH^{1/2}(\partial \Omega, \mathbb{R}^3)$, using the div-curl-gradient inequalities of vector fields, Lemma 2.1, and Sobolev imbedding theorem, we have

$$F_{H} \text{ weakly continuous on } \delta^2_{1/p}(\Omega, \text{curl}, \text{div}0, A^0).$$

After passing to another subsequence, we may assume that $A \mapsto \nabla F(x, A(x))$ is continuous in $L^p(\Omega, \mathbb{R}^3)$.

Let $v_n \rightarrow v$ weakly in $\delta^2_{1/p}(\Omega, \text{curl}, \text{div}0, A^0)$. We shall show that $\mathcal{F}[v_n, \phi_\ast] \rightarrow \mathcal{F}[v, \phi_\ast]$. Suppose not, there exist $\varepsilon > 0$ and a subsequence, still denoted by $\{v_n\}$, such that

$$|\mathcal{F}[v_n, \phi_\ast] - \mathcal{F}[v, \phi_\ast]| \geq \varepsilon. \quad (3.2)$$

After passing to another subsequence, we may assume that $v_n \rightarrow v$ strongly in $L^p(\Omega, \mathbb{R}^3)$. Hence by Lemma 3.2, $\mathcal{F}[v_n] \rightarrow \mathcal{F}[v]$. By (F2) and Proposition 3.1, for each $v_n$, the functional $\mathcal{F}[v_n, \phi]$ has a unique minimizer $\phi_{v_n}$, and

$$-c_2 \text{ meas}(\Omega) \leq \mathcal{F}[v_n, \phi_{v_n}] \leq \mathcal{F}[v_n, 0] = \mathcal{F}[v_n].$$

Hence $[\mathcal{F}[v_n, \phi_{v_n}]]$ is bounded. So we can find a subsequence such that

$$\mathcal{F}[v_{n_j}, \phi_{v_{n_j}}] \rightarrow a \quad \text{as } j \rightarrow \infty, \quad (3.3)$$

where $a$ is a constant.

Next we show that $\mathcal{F}[v, \phi_\ast] = a$. By (3.3) and (F2) we know that $\{v_{n_j} + \nabla \phi_{v_{n_j}}\}$ is bounded in $L^p(\Omega, \mathbb{R}^3)$. Since $\{v_{n_j}\}$ is bounded in $L^p(\Omega, \mathbb{R}^3)$, so $\{\nabla \phi_{v_{n_j}}\}$ is also bounded in $L^p(\Omega, \mathbb{R}^3)$, and $\{\phi_{v_{n_j}}\}$ is bounded in $W^{1, p}_0(\Omega)$. Passing to another subsequence if necessary, we may assume that $\phi_{v_{n_j}} \rightarrow \phi$ weakly in $W^{1, p}_0(\Omega)$ as $j \rightarrow \infty$. By (3.3), the convexity of $F$ and Proposition 3.1 we know that

$$a = \lim_{j \rightarrow \infty} \mathcal{F}[v_{n_j}, \phi_{v_{n_j}}] \geq \mathcal{F}[v, \phi] \geq \mathcal{F}[v, \phi_\ast]. \quad (3.4)$$

On the other hand, by (F1), and using the Taylor expansion, we have

$$\mathcal{F}[v, \phi_\ast] = \int_\Omega F(x, v_{n_j} + \nabla \phi_{v_{n_j}}) \, dx$$

$$+ \int_\Omega \nabla F(x, v_{n_j} + \nabla \phi_{v_{n_j}}) \cdot [v + \nabla \phi - (v_{n_j} + \nabla \phi_{v_{n_j}})] \, dx$$

$$+ \frac{1}{2} \int_\Omega \nabla^2 F(x, D_{n_j})[v + \nabla \phi - (v_{n_j} + \nabla \phi_{v_{n_j}})] \cdot [v + \nabla \phi - (v_{n_j} + \nabla \phi_{v_{n_j}})] \, dx, \quad (3.5)$$

where $D_{n_j} \in L^p(\Omega, \mathbb{R}^3)$ is a measurable vector field (by the Carathéodory theorem), and $D_{n_j}(x)$ is a suitable convex combination of $v(x)$ and $\nabla \phi_{v_{n_j}}(x)$ for a.e. $x \in \Omega$. From the convexity condition of $F$ we know that

$$\int_\Omega \nabla^2 F(x, D_{n_j})[v + \nabla \phi - (v_{n_j} + \nabla \phi_{v_{n_j}})] \cdot [v + \nabla \phi - (v_{n_j} + \nabla \phi_{v_{n_j}})] \, dx \geq 0.$$

Applying (3.1) to $v = v_{n_j}$, with $\psi = \phi - \phi_{v_{n_j}}$, we get

$$\int_\Omega \nabla F(x, v_{n_j} + \nabla \phi_{v_{n_j}}) \cdot \nabla (\phi - \phi_{v_{n_j}}) \, dx = 0.$$
Since \( \{v_{n_j} + \nabla \phi_{n_j}\} \) is bounded in \( L^p(\Omega, \mathbb{R}^3) \) and \( v_{n_j} \to 0 \) in \( L^p(\Omega, \mathbb{R}^3) \), using Lemma 3.2 we have
\[
\int_{\Omega} \nabla \phi(x, v_{n_j} + \nabla \phi_{n_j}) \cdot (v - v_{n_j}) \, dx \to 0, \quad \text{as } j \to \infty.
\]
So we can let \( j \to \infty \) in (3.5) and obtain \( \mathcal{F}[v, \phi_{n_j}] \geq a \). From this and (3.4) we see that \( \mathcal{F}[v, \phi] = a \). This and (3.3) contradict (3.2).

\[\square\]

### 2. Minimization of \( S_{K,G}^- \)

Now we shall verify the existence of the minimizer of the truncated functional \( S_{K,G}^- \) in \( \mathcal{S}_{\Omega}^{2,0}(\Omega, \text{curl, div0, } A^0_T) \), where \( G \) is a truncation function satisfying the condition \((G)\) in Subsection 1C.

For \( A^c_T \in TH^{1/2}(\partial \Omega, \mathbb{R}^3) \), let \( A^c \in H^1_0(\Omega, \text{div0, } A^0_T) \) be the divergence-free extension of \( A^c_T \) such that
\[
\|A^c\|_{H^1(\Omega)} \leq C(\Omega) \|A^c_T\|_{H^{1/2}(\partial \Omega)},
\]
see Ref. 18, Theorem 1.3. By the Sobolev embedding theorem we have \( A^c \in L^p(\Omega, \mathbb{R}^3) \) for any \( 1 < p < 6 \). Thus \( A^c \in \mathcal{S}_{\Omega}^{2,0}(\Omega, \text{curl, div0, } A^0_T) \). Note that any \( v \in \mathcal{S}_{\Omega}^{2,0}(\Omega, \text{curl, div0, } A^0_T) \) can be written as \( v = u + A^c \), where \( u \in \mathcal{S}_{\Omega}^{2,0}(\Omega, \text{curl, div0, } A^0_T) \). Set
\[
\mathcal{J}[u] = S_{K,G}^- [u + A^c] .
\]
Obviously
\[
\inf_{v \in \mathcal{S}_{\Omega}^-} S_G^- [v] = \inf_{u \in \mathcal{S}_{\Omega}^0} \mathcal{J}[u].
\]

Now we show that \( \mathcal{J} \) attains its minimum on \( \mathcal{S}_{\Omega}^0 \).

**Proposition 3.4.** Let \( 1 < p < 6 \) and \( 0 < K < +\infty \). Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary, and \( A^0_T \in TH^{1/2}(\partial \Omega, \mathbb{R}^3) \). Assume that \( F \) satisfies \((F1)\) and \((F2)\), and \( G \) satisfies \((G)\). Then \( \mathcal{J}[u] \) attains its minimum on \( \mathcal{S}_{\Omega}^0 \).

**Proof.** Step 1. We show that the functional \( \mathcal{J} \) is coercive in \( \mathcal{S}_{\Omega}^0 \) with equivalent norm
\[ \| \nabla u \|_{L^2(\Omega)} \]
(see Lemma 2.3). For any \( v \in \mathcal{S}_{\Omega}^{2,0}(\Omega, \text{curl, div0, } A^0_T) \) we denote
\[
\Omega_v = \{ x \in \Omega : |\text{curl } v(x)| \geq K + \delta \},
\]
where \( \delta > 0 \) is the number given in Ref. 12, Lemma 3.2, see also Lemma 2.4 above. Since \( S_K(t) > 0 \) for any \( t \in [0, +\infty) \) and \( S_K(t) = a_K t + b_1 \) for \( t \geq K + \delta \) with \( a_K, b_1 > 0 \), and \( G(t) \leq M_1 \) for all \( t \), we have
\[
S_{K,G}^- [v] \geq \int_{\Omega_v} (a_K |\text{curl } v|^2 + b_1) \, dx - M_1
\]
\[
= a_K \int_{\Omega} |\text{curl } v|^2 \, dx - a_K \int_{\Omega_v} |\text{curl } v|^2 \, dx + b_1 \text{ meas}(\Omega_v) - M_1
\]
\[
\geq a_K \int_{\Omega} |\text{curl } v|^2 \, dx - a_K (K + \delta) \text{ meas}(\Omega) - M_1.
\]
Using this and the Young’s inequality we have
\[
\mathcal{J}[u] \geq a_K \int_{\Omega} |\text{curl}(u + A^c)|^2 \, dx - a_K (K + \delta) \text{ meas}(\Omega) - M_1
\]
\[
\geq \frac{a_K}{2} \int_{\Omega} |\text{curl } u|^2 \, dx - a_K \int_{\Omega} |\text{curl } A^c|^2 \, dx - a_K (K + \delta) \text{ meas}(\Omega) - M_1.
\]
From this and applying (2.5) to \( \mathbf{u} \) then yields
\[
J[\mathbf{u}] \geq \frac{a_K}{2C(\Omega)^2} \| \nabla \mathbf{u} \|_{L^2(\Omega)}^2 - a_K \int_{\Omega} |\text{curl} \mathbf{A}^\epsilon|^2 \, dx - a_K(\Omega + \delta) \text{meas}(\Omega) - M,
\]
which shows the coercivity of \( J \) in \( Y_0 \).

Step 2. Since \( J[v, \phi_*] \) is weakly continuous in \( S^2_\rho(\Omega, \text{curl}, \text{div} 0, A^0_\Omega) \) by Lemma 3.3, the functional
\[
\mathbf{u} \mapsto J[\mathbf{u} + \mathbf{A}^\epsilon, \phi_*],
\]
is weakly continuous in \( S^2_\rho(\Omega, \text{curl}, \text{div} 0) \). This together with the convexity of the functional
\[
v \mapsto \int_{\Omega} S_K(\| \text{curl} v \|^2) \, dx,
\]
imply that \( J[\mathbf{u}] \) is weakly lower semi-continuous in \( Y_0 \). Combining this with step 1 we see that \( J \) has a minimizer \( \mathbf{u}_K \in Y_0 \). \( \square \)

Let
\[
\mathbf{v}_K = \mathbf{u}_K + \mathbf{A}^\epsilon, \quad \mathbf{A}_K = \mathbf{v}_K + \nabla \phi_{v_K}. \tag{3.7}
\]
Then \( \mathbf{v}_K \) is a minimizer of \( S^-_{K,G} \) in the set
\[
S^2_\rho(\Omega, \text{curl}, \text{div} 0, A^0_\Omega) \cap (H^1_0(\Omega) + \mathbf{A}^\epsilon).
\]
We shall derive the Euler-Lagrange equations for \( v_K \) and \( A_K \). Note that the functional \( G(J[v, \phi_*]) \) may not be differentiable in \( v \), thus we cannot derive these equations by differentiation. Here we shall borrow the method from Refs. 4 and 5.

Lemma 3.5. Under the conditions of Proposition 3.4, let \( \mathbf{v}_K \) be given in (3.7). Then for any \( \mathbf{w} \in Y_0 \), the following equality holds:
\[
\int_{\Omega} 2S_K(\| \text{curl} \mathbf{v}_K \|^2) \text{curl} \mathbf{v}_K \cdot \text{curl} \mathbf{w} \, dx - \beta_K \int_{\Omega} \nabla z F(x, \mathbf{v}_K + \nabla \phi_{v_K}) \cdot \mathbf{w} \, dx = 0, \tag{3.8}
\]
where
\[
\beta_K = G'(J[\mathbf{v}_K, \phi_{v_K}]), \quad 0 < \beta_K \leq M_2, \tag{3.9}
\]
with the positive constant \( M_2 \) given in the condition (G).

Proof. For any \( \mathbf{w} \in Y_0 \), we use the Taylor expansions for \( S_K, G, \) and \( F \) to get
\[
0 \leq J[\mathbf{u}_K + \varepsilon \mathbf{w}] - J[\mathbf{u}_K] = S^-_{K,G}[\mathbf{v}_K + \varepsilon \mathbf{w}] - S^-_{K,G}[\mathbf{v}_K]
\]
\[
= \int_{\Omega} S_K(\| \text{curl} \mathbf{v}_K + \varepsilon \text{curl} \mathbf{w} \|^2) \, dx - \int_{\Omega} S_K(\| \text{curl} \mathbf{v}_K \|^2) \, dx
\]
\[
- G(J[\mathbf{v}_K + \varepsilon \mathbf{w}, \phi_{v_K + \varepsilon \mathbf{w}}]) + G(J[\mathbf{v}_K, \phi_{v_K}])
\]
\[
= \varepsilon \int_{\Omega} 2S_K(\| \text{curl} \mathbf{v}_K \|^2) \text{curl} \mathbf{v}_K \cdot \text{curl} \mathbf{w} \, dx + \epsilon^2 \int_{\Omega} S''_K(\| \text{curl} \mathbf{v}_K \|^2) \| \text{curl} \mathbf{w} \|^2 \, dx
\]
\[
+ \frac{1}{2} \epsilon^2 \int_{\Omega} S''(t, \mathbf{v}_K) \left( 2 \text{curl} \mathbf{v}_K \cdot \text{curl} \mathbf{w} + \epsilon \| \text{curl} \mathbf{w} \|^2 \right) \, dx
\]
\[
- G'(\xi) \int_{\Omega} \left( \nabla z F(x, \mathbf{v}_K + \nabla \phi_{v_K}, \varepsilon \mathbf{w} + \nabla \phi_{v_K + \varepsilon \mathbf{w}} - \nabla \phi_{v_K}) \right) \, dx
\]
\[
- \frac{1}{2} G'(\xi) \int_{\Omega} \left( \nabla^2 z F(x, D_z(\xi)) \left( \varepsilon \mathbf{w} + \nabla \phi_{v_K + \varepsilon \mathbf{w}} - \nabla \phi_{v_K} \right) \right) \, dx.
\]
In the above,

\[ t_\varepsilon(x) = \alpha_\varepsilon(x) \left| \text{curl} \ v_K(x) \right|^2 + (1 - \alpha_\varepsilon(x)) \left| \text{curl} \ v_K(x) + \varepsilon \text{curl} \ w(x) \right|^2, \]

\[ \xi_\varepsilon = \gamma_\varepsilon \mathcal{F}[v_K, \phi_{v_k}] + (1 - \gamma_\varepsilon)\mathcal{F}[v_K + \varepsilon w, \phi_{v_k + \varepsilon w}], \]

where \( \alpha_\varepsilon(x) \) is a measurable function and \( 0 \leq \alpha_\varepsilon \leq 1 \) for a.e. \( x \in \Omega \), \( \gamma_\varepsilon \) is a constant and \( 0 \leq \gamma_\varepsilon \leq 1 \). \( D_\varepsilon(x) \in L^p(\Omega, \mathbb{R}^3) \) is also a measurable vector field which is a suitable convex combination of

\[ v_K(x) + \nabla \phi_{v_k}(x) \quad \text{and} \quad v_K(x) + \varepsilon w(x) + \nabla \phi_{v_k + \varepsilon w}(x) \]

for a.e. \( x \in \Omega \).

Applying (3.1) with \( v = v_K \) and \( \psi = \phi_{v_k + \varepsilon w} - \phi_{v_k} \), we obtain

\[ \int_{\Omega} \left( \nabla F(x, v_K + \nabla \phi_{v_k}), \nabla \phi_{v_k + \varepsilon w} - \nabla \phi_{v_k} \right) \, dx = 0. \] 

(3.11)

Using the convexity of \( F \) and the condition (G) we see that the last term in the right side of (3.10) is non-positive. We drop this term, substitute (3.11) into (3.10), and then cancel \( \varepsilon \) from both sides to get

\[
0 \leq \int_{\Omega} 2S_K'(|\text{curl} v_K|^2) \text{curl} v_K \cdot \text{curl} w \, dx + \varepsilon \int_{\Omega} S_K'(|\text{curl} v_K|^2) |\text{curl} w|^2 \, dx \\
+ \frac{\varepsilon}{2} \int_{\Omega} S_K''(\xi_\varepsilon) \left( 2 \text{curl} v_K \cdot \text{curl} w + \varepsilon |\text{curl} w|^2 \right)^2 \, dx \\
- G'(\xi_\varepsilon) \int_{\Omega} \left| \nabla F(x, v_K + \nabla \phi_{v_k}) \right| \, dx.
\]

(3.12)

After passing to a subsequence we may assume that \( \gamma_\varepsilon \to \gamma \) as \( \varepsilon \to 0 \), where \( 0 \leq \gamma \leq 1 \). By Lemma 3.3 we know that

\[ \mathcal{F}[v_K + \varepsilon w, \phi_{v_k + \varepsilon w}] \to \mathcal{F}[v_K, \phi_{v_k}] \quad \text{as} \quad \varepsilon \to 0. \]

Hence \( \xi_\varepsilon \to \mathcal{F}[v_K, \phi_{v_k}] \) as \( \varepsilon \to 0 \). Thus by (G) we have \( G'(\xi_\varepsilon) \to G'(\mathcal{F}[v_K, \phi_{v_k}]) \) as \( \varepsilon \to 0 \). Letting \( \varepsilon \to 0 \) in (3.12) we obtain

\[
0 \leq \int_{\Omega} 2S_K'(|\text{curl} v_K|^2) \text{curl} v_K \cdot \text{curl} w \, dx \\
- G'(\mathcal{F}[v_K, \phi_{v_k}]) \int_{\Omega} \nabla F(x, v_K + \nabla \phi_{v_k}) \cdot w \, dx.
\]

Replacing \( w \) by \( -w \) we deduce that for any \( w \in \mathcal{H}_0 \),

\[
0 = \int_{\Omega} 2S_K'(|\text{curl} v_K|^2) \text{curl} v_K \cdot \text{curl} w \, dx \\
- G'(\mathcal{F}[v_K, \phi_{v_k}]) \int_{\Omega} \nabla F(x, v_K + \nabla \phi_{v_k}) \cdot w \, dx.
\]

Denote \( \beta_K = G'(\mathcal{F}[v_K, \phi_{v_k}]) \). From (G) we see that \( 0 < \beta_K \leq M_2 \). Now (3.8) and (3.9) follow. \( \square \)

Let us mention that the constant \( \beta_K \) in (3.8) depends not only on \( K, A_K^0 \), but also on the solution \( A_K \).

Now we show that \( A_K \) satisfies the following:

\[
\int_{\Omega} 2S_K'(|\text{curl} A|^2) \text{curl} A \cdot \text{curl} H \, dx - \beta_K \int_{\Omega} \nabla F(x, A) \cdot H \, dx = 0, \quad \forall H \in \mathcal{H}_2^+(\Omega). \] 

(3.13)

Proposition 3.6. Under the conditions of Proposition 3.4, let \( A_K \in \mathcal{H}_2^+(\Omega, \text{curl, } A_K^0) \) be given in (3.7). Then \( A_K \) satisfies (3.13) for all \( H \in \mathcal{H}_2^+(\Omega) \), where \( \beta_K \) is given in (3.9).
Proof. For any \( H \in C^1_0(\Omega, \mathbb{R}^3) \cap \mathbb{H}_2^\bot(\Omega) \), we decompose it into \( H = w + \nabla \psi \), where \( \psi \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) is the solution of
\[
\Delta \psi = \text{div}H \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial \Omega,
\]
and \( w \in \mathfrak{g}_0 \). Then (3.8) holds for this test field \( w \). Applying (3.1) to \( v = v_K \) with this \( \psi \) as a test function, we get
\[
\int_\Omega \nabla_z F(x, v_K + \nabla \phi_K) \cdot \nabla \psi \, dx = 0.
\]
Adding this equality to (3.8) we obtain
\[
\int_\Omega 2S'_K(|\text{curl } v_K|^2) \text{curl } v_K \cdot \text{curl } H \, dx - \beta_K \int_\Omega \nabla_z F(x, v_K + \nabla \phi_K) \cdot H \, dx = 0.
\]
Note that \( \text{curl}(\nabla \phi_K) = 0 \) holds in the distribution sense (see Ref. 13), and by the representation formula of \( H^{-1}(\Omega, \mathbb{R}^3) \), we know that this is also true in \( L^2(\Omega, \mathbb{R}^3) \). Hence in the first term in the above equality we can replace \( \text{curl } v_K \) by \( \text{curl } A_K \), and get, for any \( H \in C^1_0(\Omega, \mathbb{R}^3) \cap \mathbb{H}_2^\bot(\Omega) \),
\[
\int_\Omega 2S'_K(|\text{curl } A_K|^2) \text{curl } A_K \cdot \text{curl } H \, dx - \beta_K \int_\Omega \nabla_z F(x, A_K) \cdot H \, dx = 0. \tag{3.14}
\]
Since \( C^1_0(\Omega, \mathbb{R}^3) \cap \mathbb{H}_2^\bot(\Omega) \) is dense in \( \mathbb{H}_2^\bot(\Omega) \), (3.14) holds for all \( H \in \mathbb{H}_2^\bot(\Omega) \). Hence \( A_K \) satisfies (3.13).

Using the structure of the space \( \mathbb{H}_2(\Omega) \) (see Ref. 13, p. 222), we may view (3.13) as a weak formula of the system
\[
\begin{align*}
\text{curl } (S'_K(|\text{curl } A|^2) \text{curl } A) - \frac{\beta_K}{2} \nabla_z F(x, A) &= \nabla \Phi & \text{in } \Omega, \\
A_T &= A_T^0 & \text{on } \partial \Omega,
\end{align*}
\]
where \( \nabla \Phi \in \mathbb{H}_2(\Omega) \) is a vector field, and \( \beta_K \) is a constant, both being determined by the solution \( A_K \). If \( \nabla \Phi = 0 \), then the above equation is reduced to (1.24).

Corollary 3.7. Under the conditions of Proposition 3.4, if furthermore assume that \( \Omega \) has no holes, then \( A_K \in \mathcal{S}^2_\Omega(\Omega) \), \( \text{curl}, A_T^0 \) given in (3.7) is a weak solution of system (1.24) with \( \beta = \beta_K \).

Proof. If \( \Omega \) has no holes, then \( \mathbb{H}_2(\Omega) = \{0\} \) and \( \mathbb{H}_2^\bot(\Omega) = L^2(\Omega, \mathbb{R}^3) \). Then from the proof of Proposition 3.6 we see that (3.14) holds for all \( H \in C^1_0(\Omega, \mathbb{R}^3) \). Thus \( A_K \in \mathcal{S}^2_\Omega(\Omega) \), \( \text{curl}, A_T^0 \) is a weak solution of system (1.24).

B. Estimates of the critical points of \( \mathcal{S}^G_{K,G} \)

In this subsection we look for \( L^2 \) and \( C^{2,a} \) estimates of the weak solutions of (1.24) which have been obtained in last subsection. We shall only consider the special case where
\[
F(x, z) = a(x)|z|^2, \tag{3.15}
\]
where \( a(x) \) satisfies the condition (A2). Obviously this \( F \) satisfies (F1) and (F2) with \( p = 2 \). Let \( 0 < K < +\infty \). According to the discussion in Subsection III A we know that, if \( \Omega \) has no holes, then \( \mathcal{S}^G_{K,G} \) has a critical point \( A_K \in \mathcal{S}^2_\Omega(\Omega) \), \( \text{curl}, A_T^0 \), and it is a solution of the Euler-Lagrange equation
\[
\begin{align*}
\text{curl } (S'_K(|\text{curl } A|^2) \text{curl } A) - \beta_K a(x)A &= 0 & \text{in } \Omega, \\
A_T &= A_T^0 & \text{on } \partial \Omega.
\end{align*}
\tag{3.16}
\]
Recall that $A_K = v_K + \nabla \phi_{v_K}$ with $v_K \in S_T^{1,2}(\Omega, \curl, \div, A_0^T)$ and $\phi_{v_K} \in H_0^1(\Omega)$, where $v_K$ is a weak solution of
\[
\begin{aligned}
\curl \left( S' \left( \curl v \right) \right) \curl v - \beta_K a(x) \left( v + \nabla \phi_{v_K} \right) &= 0 & \text{in } \Omega, \\
\n
v_T &= A_0^T & \text{on } \partial \Omega,
\end{aligned}
\]  
(3.17)
and $\phi_{v_K}$ is a weak solution of
\[
\begin{aligned}
\div \left( a(x) \left( v + \nabla \phi \right) \right) &= 0 & \text{in } \Omega, \\
\phi &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]  
(3.18)
Now we derive the $L^2$ estimate of $v_K$ and $\nabla \phi_{v_K}$. Define a positive constant
\[
N_1(\Omega, a) = 16 \| a \|_{C(\Omega)} C_{vp}(\Omega)^2 \left[ 2 + (1 + C_p(\Omega)) a_0^{-1} \| a \|_{C(\Omega)} \right],
\]
where $C_p(\Omega)$ and $C_{vp}$ are the constants in the Poincaré inequality (2.3) and in the variation of Poincaré inequality (2.4), respectively. As mentioned in Subsection 1C, we can always choose the function $G$ to satisfy (G) and also
\[
M_2 \equiv \sup_{r \in \mathbb{R}} \| G(t) \| \leq \frac{\min S' \left( \curl \right)}{N_1(\Omega, a)}.
\]  
(3.19)
As an immediate consequence, the constant $\beta_K$ in (3.16) satisfies
\[
0 < \beta_K \leq \frac{\min S' \left( \curl \right)}{N_1(\Omega, a)}.
\]  
(3.20)

**Lemma 3.8.** Let $0 < K < +\infty$. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, and $A_0^T \in \text{TH}^{1/2}(\partial \Omega, \mathbb{R}^3)$. Assume that $a(x) \in C(\overline{\Omega})$ is a positive function, and $G$ is chosen to satisfy (G) and (3.19). Let $v_K$ and $\phi_{v_K}$ be the weak solution of (3.17) and (3.18), respectively. Then
\[
\| v_K \|_{H^1(\Omega)} + \| \phi_{v_K} \|_{H^1(\Omega)} \leq C \| A_0^T \|_{H^{1/2}(\partial \Omega)},
\]  
(3.21)
where $C$ depends on $\Omega$, $a_0$, $\| a \|_{C(\Omega)}$, $\beta_K$, and $K$.

**Proof.** In the proof, for simplicity of notation we shall drop the subscripts and write $v_K$, $\phi_{v_K}$, and $\beta_K$ as $v$, $\phi$, and $\beta$, and denote $C_p(\Omega)$ and $C_{vp}(\Omega)$ by $C_p$ and $C_{vp}$, respectively.

**Step 1.** Since $\phi \in H_0^1(\Omega)$ is a weak solution of (3.18), we have
\[
\| \nabla \phi \|_{L^2(\Omega)} \leq C_0^{-1} \| a \|_{C(\Omega)} \| v \|_{L^2(\Omega)}.
\]
Using the Poincaré inequality (2.3), we obtain
\[
\| \phi \|_{H^1(\Omega)} \leq C_3 \| v \|_{L^2(\Omega)},
\]  
(3.22)
where $C_3 = (1 + C_p) a_0^{-1} \| a \|_{C(\Omega)}$.

**Step 2.** Let us denote by $A^e$ the divergence-free extension of $A_0^T$ satisfying (3.6). Write $v = u + A^e$. Then $u \in S_T^{0,\rho}(\Omega, \curl, \div)$. Taking $u$ as a test field for the weak form of (3.17), and using the Cauchy inequality we get
\[
\| \curl v \|_{L^2(\Omega)}^2 \leq \frac{\max S' \left( \curl \right)}{\min S' \left( \curl \right)} \| \curl A^e \|_{L^2(\Omega)}^2 + \frac{2\beta a \| a \|_{C(\Omega)}}{\min S' \left( \curl \right)} \left[ (2 + C_3) \| v \|_{L^2(\Omega)}^2 + \| A^e \|_{L^2(\Omega)}^2 \right].
\]  
(3.23)
Using (2.4) we get
\[
\| u \|_{L^2(\Omega)} \leq C_{vp} \| \curl u \|_{L^2(\Omega)} \leq C_{vp} \left( \| \curl v \|_{L^2(\Omega)} + \| \curl A^e \|_{L^2(\Omega)} \right),
\]
hence
\[
\| v \|_{L^2(\Omega)} \leq C_{vp} \left( \| \curl v \|_{L^2(\Omega)} + \| \curl A^e \|_{L^2(\Omega)} \right) + \| A^e \|_{L^2(\Omega)}.
\]
From this and (3.23) we have
\[
\|v\|_{L^2(\Omega)}^2 \leq 8(2 + C_3)C_{vp}^2 \frac{\beta \|a\|_{C^2(\bar{\Omega})}}{\min S_K'} \|v\|_{L^2(\Omega)}^2 + 4C_{vp}^2 \left(1 + \frac{\max S_K'}{\min S_K'}\right) \|\text{curl} A'\|_{L^2(\Omega)}^2
\]
\[
+ 2\left[4(2 + C_3)C_{vp}^2 \frac{\beta \|a\|_{C^2(\bar{\Omega})}}{\min S_K'} + 1\right] \|A'\|_{L^2(\Omega)}^2.
\]  
(3.24)

From (3.20) we see that
\[
\beta \|a\|_{C^2(\bar{\Omega})} \leq \frac{\min S_K'}{16(2 + C_3)C_{vp}^2}.
\]
So we get from (3.24)
\[
\|v\|_{L^2(\Omega)}^2 \leq 8C_{vp}^2 \left(1 + \frac{\max S_K'}{\min S_K'}\right) \|\text{curl} A'\|_{L^2(\Omega)}^2
\]
\[
+ 4\left[4(2 + C_3)C_{vp}^2 \frac{\beta \|a\|_{C^2(\bar{\Omega})}}{\min S_K'} + 1\right] \|A'\|_{L^2(\Omega)}^2.
\]  
(3.21) follows from this, (3.6), (3.22), and Ref. 13, Corollary 1, p. 212 immediately. \(\square\)

The \(C^{2,\alpha}\) estimate of \(v_K\) can be derived as in the proof of Proposition 4.5 in Ref. 12, with \(a(x)\) there replaced by \(-\beta_K a(x)\).

**Proposition 3.9.** Let \(0 < K < b^+.\) Assume that \(\Omega, a(x), A_T^0\) satisfy (A1), (A2), and (A3), respectively with \(0 < \alpha < 1\), and \(G\) is chosen to satisfy (G) and (3.19). Let \(v_K\) and \(\phi_K\) be the weak solution of (3.17) and (3.18), respectively. Then \(v_K \in C^{2,\alpha}(\bar{\Omega}, \text{div}0, A_T^0), \phi_K \in C^{2,\alpha}(\bar{\Omega}),\) and
\[
\|v_K\|_{C^{2,\alpha}(\Omega)} + \|\nabla \phi_K\|_{C^{1,\alpha}(\Omega)} \leq C,
\]
where \(C\) depends on \(\Omega, \alpha, a_0, \|a\|_{C^{1,\alpha}(\bar{\Omega})}, K, \delta, M_2,\) and \(\|A_T^0\|_{C^{1,\alpha}(\bar{\Omega})}.
\)

Let us mention that we can furthermore require
\[
S_K \in C^4 ([0, +\infty)), \quad f_K \in C^3 ([0, +\infty)).
\]
In this case we have

**Corollary 3.10.** In addition to the conditions of Theorem 3.9, assume furthermore that \(a(x) \in C^{2,\alpha}(\bar{\Omega}).\) Then
\[
A_K = v_K + \nabla \phi_K \in C^{3,\alpha}(\bar{\Omega}, \text{div}0, A_T^0) + \text{grad} C^{3,\alpha}_0(\bar{\Omega}) \subset C^{2,\alpha}(\bar{\Omega}, \mathbb{R}^3),
\]
and
\[
\|v_K\|_{C^{3,\alpha}(\Omega)} + \|\nabla \phi_K\|_{C^{2,\alpha}(\Omega)} \leq C,
\]
where \(C\) depends on \(\Omega, \alpha, a_0, \|a\|_{C^{1,\alpha}(\bar{\Omega})}, K, \delta, M_2,\) and \(\|A_T^0\|_{C^{2,\alpha}(\bar{\Omega})}.
\)

### C. Solutions of the Dirichlet problem (1.8)

In Subsections IIIA and IIIB we have proved the existence and regularity of the solutions \(A_K\) of the modified system (1.24). In this subsection, for \(F(x, A)\) given in (3.15) we shall show the smallness of \(\|\text{curl} A_K\|_{L^\infty(\Omega)}\) for small boundary datum \(A_T^0\), hence \(A_K\) is exactly the classical solution of the Dirichlet problem (1.8). For this purpose we need the first eigenvalue of the following problem
\[
\text{curl}^2 u = \lambda u \quad \text{in} \ \Omega, \quad u_T = 0 \quad \text{on} \ \partial \Omega.
\]  
(3.25)

We say that \(u\) is an eigenvector of (3.25) associated with an eigenvalue \(\lambda\) if \(u \in H^2_{T_0}(\Omega, \text{curl}), u \neq 0\) and it is a weak solution of (3.25).
Lemma 3.11. Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary and without holes. Then the first eigenvalue \( \lambda_1 \) of (3.25) is positive and is given by

\[
\lambda_1 = \min_{u \in H_0^1(\Omega, \text{curl}, \text{div}0), \ u \neq 0} \frac{\| \text{curl} u \|^2_{L^2(\Omega)}}{\| u \|^2_{L^2(\Omega)}}.
\]  

(3.26)

Proof. Obviously the number \( \lambda_1 \) defined by (3.26) is the lowest eigenvalue. Now we show \( \lambda_1 > 0 \). Otherwise, there exists \( u \in H_0^1(\Omega, \text{curl}, \text{div}0), \ u \neq 0 \), such that \( \text{curl} u = 0 \). Then \( u \in H_2(\Omega) \). But \( H_2(\Omega) = \{0\} \) since \( \Omega \) has no holes. Thus \( u = 0 \), a contradiction. \( \square \)

For the given positive function \( a(x) \) with \( a_0 = \min_{x \in \Omega} a(x) > 0 \), we define

\[
N_2(\Omega, a) = 2\| a \|_{C(\overline{\Omega})} \left(1 + a_0^{-2}\| a \|^2_{C(\overline{\Omega})}\right).
\]  

(3.27)

As mentioned before we can always choose the function \( G \) to be small. So we assume \( G \) satisfies (G), (3.19), and

\[
M_2 < \frac{\lambda_1 \min S^0}{N_2(\Omega, a)}.
\]  

(3.28)

Proposition 3.12. Let \( 0 < K < b^2 \). Assume that \( \Omega \) and \( a(x) \) satisfy (A1) and (A2), respectively, with \( 0 < \alpha < 1 \), and \( G \) is chosen to satisfy (G), (3.19), and (3.28). Let

\[
A_K = v_K + \nabla \phi_K \in C_t^2(\overline{\Omega}), \text{div}0, A_T^0 + \text{grad} C_0^2(\overline{\Omega})
\]

be the solution of system (3.16) obtained in Proposition 3.6. Then for any \( \varepsilon > 0 \) and any \( \sigma \in [0, \alpha) \), there exists a constant \( \eta = \eta(\varepsilon, \sigma) > 0 \) such that if \( A_T^0 \) satisfies (A3) with

\[
\| A_T^0 \|_{C^2(\Omega)} \leq \eta,
\]

then we have \( \| A \|_{C^1(\Omega)} \leq \varepsilon \).

Proof. Suppose the conclusion were false. Then there exists \( 0 < \sigma < \alpha, \varepsilon_0 > 0 \) and a sequence \( \{A_T^0\} \) with each of them satisfying (A3) and \( \| A_T^0 \|_{C^1(\Omega)} \to 0 \), and for each \( n \) the solution \( A_{K,n} = v_{K,n} + \nabla \phi_{K,n} \) of (3.16) with the constant \( \beta_K = \beta_{K,n} \), such that

\[
\| A_{K,n} \|_{C^1(\Omega)} \geq \varepsilon_0.
\]

For simplicity, in the proof we denote \( A_{K,n}, v_{K,n}, \phi_{K,n}, \) and \( \beta_{K,n} \) by \( A_n, v_n, \phi_n, \) and \( \beta_n \), respectively.

From \( (G) \) we know that \( 0 < \beta_n \leq M_2 \) for all \( n \). According to Proposition 3.9, \( \| v_n \|_{C^2(\Omega)} \), \( \| \nabla \phi_n \|_{C^1(\Omega)} \) and \( \| A_n \|_{C^1(\Omega)} \) are uniformly bounded in \( n \). We choose subsequences of \( \{v_n\} \) and \( \{\phi_n\} \), denoted as \( \{v_n\} \) and \( \{\phi_n\} \), such that as \( j \to \infty \),

\[
\| A_{n_j} \|_{C^1(\Omega)} \to \limsup_{n \to \infty} \| A_n \|_{C^1(\Omega)}.
\]

After passing to a subsequence again, we may also assume that

\[
\begin{align*}
v_{n_j} & \to v \quad \text{in} \ C^2(\Omega, \mathbb{R}^3), & v_T = 0 \quad \text{on} \ \partial \Omega, \\
\phi_{n_j} & \to \phi \quad \text{in} \ C^2(\Omega, \mathbb{R}^3), & \phi = 0 \quad \text{on} \ \partial \Omega, & (3.29) \\
\beta_{n_j} & \to \beta \leq M_2.
\end{align*}
\]

Hence \( A_{n_j} \to A \equiv v + \nabla \phi \) in \( C^1(\Omega, \mathbb{R}^3) \) and \( A_T = 0 \) on \( \partial \Omega \), and

\[
\| A \|_{C^1(\Omega)} \geq \varepsilon_0.
\]  

(3.30)

Recall the integral form of the equation for \( A_{n_j} \): For any \( H \in C_0^1(\tilde{\Omega}, \mathbb{R}^3) \),

\[
\int_{\Omega} S'_K(\| \text{curl} A_{n_j} \|^2) \text{curl} A_{n_j} \cdot \text{curl} H \, dx - \beta_{n_j} \int_{\Omega} a(x) A_{n_j} \cdot H \, dx = 0.
\]
Letting \( j \to \infty \) and then setting \( \mathbf{H} = \mathbf{A} \) as the test field in the resulted equality, we find
\[
(\min S'_K) \| \text{curl} \mathbf{v} \|^2_{L^2(\Omega)} = (\min S'_K) \| \text{curl} \mathbf{A} \|^2_{L^2(\Omega)} \leq \beta \| a \|_{C(\bar{\Omega})} \| \mathbf{A} \|^2_{L^2(\Omega)}.
\] (3.31)

Similarly, \( \phi_{\nu_j} \) satisfies, for any \( \psi \in H^1_0(\Omega) \),
\[
\int_{\Omega} a(x)(\mathbf{v}_{\nu_j} + \nabla \phi_{\nu_j}) \cdot \nabla \psi \, dx = 0.
\]

Letting \( j \to \infty \) and then setting \( \psi = \phi \) as the test function in the resulted equality we deduce that
\[
\| \nabla \phi \|_{L^2(\Omega)} \leq a_0^{-1} \| a \|_{C(\bar{\Omega})} \| \mathbf{v} \|_{L^2(\Omega)}.
\] (3.32)

Hence we have
\[
\| \mathbf{A} \|^2_{L^2(\Omega)} \leq 2 \left( \| \mathbf{v} \|^2_{L^2(\Omega)} + \| \nabla \phi \|^2_{L^2(\Omega)} \right) \leq 2 \left( 1 + a_0^{-2} \| a \|^2_{C(\bar{\Omega})} \right) \| \mathbf{v} \|^2_{L^2(\Omega)}.
\] (3.33)

From (3.31) and (3.33) we get
\[
\| \text{curl} \mathbf{v} \|^2_{L^2(\Omega)} \leq \frac{2\beta}{\min S'_K} \| a \|_{C(\bar{\Omega})} \left( 1 + a_0^{-2} \| a \|^2_{C(\bar{\Omega})} \right) \| \mathbf{v} \|^2_{L^2(\Omega)}.
\] (3.34)

Using (3.29), (3.27), and (3.28) we find that
\[
\frac{2\beta}{\min S'_K} \| a \|_{C(\bar{\Omega})} \left( 1 + a_0^{-2} \| a \|^2_{C(\bar{\Omega})} \right) < \lambda_1.
\]

From this and (3.34), and by the definition of \( \lambda_1 \), we know that \( \mathbf{v} = \mathbf{0} \). Then from (3.33) we have \( \mathbf{A} = \mathbf{0} \), which contradicts (3.30).

**Proof of Theorem 1.1.** Let us fix a constant \( K \) such that \( 0 < K < b^2 \) and construct the function \( S_K \) as in Subsection II D. Then we find a function \( G \) satisfying (\( G \)), (3.19), and (3.28). From Proposition 3.6, for any \( \mathbf{A}_0^\beta \) satisfying (A3), the system (3.16) with this \( S_K \) has a weak solution \( \mathbf{A}_K = \mathbf{v}_K + \nabla \phi_{\nu_K} \).

From Proposition 3.9 we know that \( \mathbf{A}_K \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^3) \). From Proposition 3.12 we know that there exists \( \eta > 0 \) such that if \( \| A_0^\beta \|_{C^{2,\alpha}(\bar{\Omega})} \leq \eta \), then \( \| \mathbf{A}_K \|_{C^{2,\alpha}(\bar{\Omega})} \leq \sqrt{K} \). Hence
\[
S'_K (\| \text{curl} \mathbf{A}_K(x) \|^2) = S'(\| \text{curl} \mathbf{A}_K(x) \|^2), \quad \forall x \in \Omega.
\]

Therefore, \( \mathbf{A}_K \) is a solution of (1.8). Since \( K \) is fixed now, we can denote \( \mathbf{A}_K \) by \( \mathbf{A} \) and \( \beta_K \) given in (3.9) by \( \beta \). Then \( \mathbf{A} \) is a solution of (1.8) with the constant \( \beta \).

In Subsection III A we obtained critical points for the truncated functionals with fairly general nonlinear functions \( F \). In order to show that they are solutions of the extended magnetostatic Born-Infeld systems we need to prove the \( C^{1,\alpha} \) regularity and uniform estimates of the solution, which are established in Theorem 1.1 for the special form of \( F \). For the system with a general function \( F \), in order to overcome the difficulties in proving the \( C^{1,\alpha} \) regularity, one may try to modify the function \( F(x, z) \) and apply the method in this paper to deal with the modified functional. We wish to examine this approach in the future.

**IV. THE NEUMANN PROBLEM**

In Subsection IV A we assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^4 \) boundary,
\[
\mathbf{D}_g^0 \in TW^{1-1/r',r'}(\partial \Omega, \mathbb{R}^3), \quad g \in H^{1/2}(\partial \Omega),
\] (4.1)

\( F \) satisfies (F1) and (F2), \( 1 < p < 6, \quad r = \min\{2, p\} \).
A. Critical points of $\mathcal{H}_{K,G}$

1. Minimization of $F^c$

Let $F^c$ be the functional defined in (1.25).

Lemma 4.1. Assume that $\Omega$, $D_T^0$, $g$, $F$, $p$, $r$ satisfy (4.1). Then for any $v \in \mathcal{S}^2_p(\Omega, \text{curl}, \text{div}0, g)$ the functional $F^c[v, \cdot]$ has a unique minimizer $\phi^c \in \mathcal{W}^{1,p}(\Omega)$.

Proof. Using the condition on $p$, $g$ in (4.1) and Lemma 2.1 we see that

$$H^1_0(\Omega, \text{div}0, g) = \{v \in H^1(\Omega, \mathbb{R}^3) : \text{div}v = 0 \text{ in } \Omega, \ v \cdot v = g \text{ on } \partial\Omega\}. $$

We also know that $H^1_0(\Omega, \text{div}0, g)$ is compactly imbedded into $L^p(\Omega, \mathbb{R}^3)$. If $\phi \in \mathcal{W}^{1,p}(\Omega)$, then $\nabla \phi \in \mathcal{S}^2_p(\Omega, \text{curl})$, thus $(\nabla \phi)_T \in W^{-1/r,r}(\partial\Omega, \mathbb{R}^3)$. Using the condition on $D_T^0$ in (4.1) we see that the boundary integral part of $F^c$ is well defined for $\phi \in \mathcal{W}^{1,p}(\Omega)$. Thus for any $v \in \mathcal{S}^2_p(\Omega, \text{curl}, \text{div}0, g)$, the functional $F^c[v, \cdot]$ is well-defined on $\mathcal{W}^{1,p}(\Omega)$. Denote

$$a(F^c, v) = \inf_{\phi \in \mathcal{W}^{1,p}(\Omega)} F^c[v, \phi].$$

Let $\{\phi_n\} \subset \mathcal{W}^{1,p}(\Omega)$ be a minimizing sequence. Using the Cauchy inequality we have

$$\left| \int_{\partial\Omega} (D_T^0 \times (\nabla \phi_n)_T) \cdot v \ dS \right| \leq \|D_T^0\|_{W^{-1/r',r'}(\partial\Omega)} \|\nabla \phi_n\|_{W^{-1/r,r}(\partial\Omega)}$$

$$\leq \frac{\epsilon C(p, \Omega)}{2} \|\nabla \phi_n\|_{L^p(\Omega)} + \frac{C(\epsilon)}{2} \|D_T^0\|_{W^{-1/r',r'}(\partial\Omega)},$$

where $\epsilon$ is a small positive constant to be determined later. This together with (F2) yields

$$a(F^c, v) + o(1) \geq \int_{\Omega} F(x, v + \nabla \phi_n) \ dx - 2 \int_{\partial\Omega} (D_T^0 \times (\nabla \phi_n)_T) \cdot v \ dS$$

$$\geq \int_{\Omega} (2^{-p}c_1|\nabla \phi_n|^p - c_1|v|^p - c_2) \ dx - \epsilon C(p, \Omega)\|\nabla \phi_n\|_{L^p(\Omega)} - C(\epsilon)\|D_T^0\|_{W^{-1/r',r'}(\partial\Omega)}.$$ 

By choosing $\epsilon = 2^{-p - 1}c_1/C(p, \Omega)$ we see that $\{\nabla \phi_n\}$ is bounded in $L^p(\Omega, \mathbb{R}^3)$, hence $\{\phi_n\}$ is bounded in $\mathcal{W}^{1,p}(\Omega)$. Since $\mathcal{W}^{1,p}(\Omega)$ is compactly imbedded in $L^p(\Omega)$ and $\mathcal{W}^{1,1}(\Omega)$ is weakly closed, there exists a subsequence $\{\phi_{n_k}\}$ and $\phi^c \in \mathcal{W}^{1,p}(\Omega)$ such that $\phi_{n_k} \rightharpoonup \phi^c$ weakly in $\mathcal{W}^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ as $k \to +\infty$. By the strict convexity of $F(x, z)$ in $z$ we see that $\phi^c$ is the unique minimizer of $F^c[v, \cdot]$. \hfill $\Box$

Now we show that the Euler-Lagrange equation of $\phi^c$ is (1.26). Since $\phi^c$ is a minimizer of $F^c[v, \cdot]$ in $\mathcal{W}^{1,p}(\Omega)$, a direct computation shows that for any $\psi \in \mathcal{C}^1(\tilde{\Omega})$,

$$\int_{\Omega} \nabla \psi \cdot F(x, v + \nabla \phi^c) - 2 \int_{\partial\Omega} (D_T^0 \times (\nabla \psi)_T) \cdot v \ dS = 0. \quad (4.2)$$

For any $\psi \in \mathcal{C}^1(\tilde{\Omega})$ there exists a constant $c$ such that $\psi \in \mathcal{C}^1(\tilde{\Omega})$, and hence (4.2) holds for all $\psi \in \mathcal{C}^1(\Omega)$. Let $D_T^c \in H^1(\Omega, \text{div}0, D_T^0)$ be the divergence-free extension of $D_T^0$. Using integration by parts we get

$$\int_{\partial\Omega} (D_T^c \times (\nabla \psi)_T) \cdot v \ dS = \int_{\Omega} \text{curl} D_T^c \cdot \nabla \psi \ dx = \int_{\partial\Omega} (v \cdot \text{curl} D_T^0) \psi \ dS, \quad (4.3)$$
where in the last equality we have used the identity \( \nu \cdot \text{curl} \, D^\nu = \nu \cdot \text{curl} \, D_T^\nu \) on \( \partial \Omega \). From (4.2) and (4.3) we obtain, for all \( \psi \in C^1(\Omega) \), and hence for all \( \psi \in W^{1,p}(\Omega) \),

\[
\int_{\Omega} \nabla_z F(x, \nu + \nabla \phi^z) \cdot \nabla \psi \, dx - 2 \int_{\partial \Omega} (\nu \cdot \text{curl} \, D_T^\nu) \psi \, dS = 0.
\]

Therefore \( \phi^z \) is a weak solution of (1.26).

**Remark 4.2.** If \( F(x, z) = b(x, |z|^2) \), where \( b(x, s) \) is \( C^1 \) in \( s \) and \( \partial_s b(x, s) \neq 0 \) for \( x \in \partial \Omega, \ s \in \mathbb{R} \), and if \( \nu \cdot \text{curl} \, D_T^\nu = 0 \) on \( \partial \Omega \), then for any \( \nu \in \mathcal{S}_{n+2}(\Omega, \text{curl}, \text{div}0, g) \) Eq. (1.26) can be written as

\[
\begin{cases}
\partial_t b(x, |\nu + \nabla \phi|^2)(\nu + \nabla \phi) = 0 & \text{in } \Omega, \\
\frac{\partial \phi^z}{\partial \nu} = -g & \text{on } \partial \Omega.
\end{cases}
\]

Arguing as in the proof of Lemma 3.3 we can prove the following conclusion.

**Lemma 4.3.** Assume that \( \Omega, D_T^0, g, F, p, r \) satisfy (4.1). Then the functional \( \mathcal{I}^r[\nu, \phi] \) is continuous on \( \mathcal{S}_{n+2}(\Omega, \text{curl}, \text{div}0, g) \times W^{1,p}(\Omega) \), and the functional

\[
\nu \mapsto \mathcal{I}^r[\nu, \phi^z]
\]

is weakly continuous in \( \mathcal{S}_{n+2}(\Omega, \text{curl}, \text{div}0, g) \).

**2. Minimization of \( \mathcal{I}^r_{K,G} \)**

Now we consider the functional \( \mathcal{I}^r_{K,G} \) introduced in (1.27), where \( G \) is a truncation function satisfying (G). Due to the non-coercivity of the leading order term of the functional \( \mathcal{I}^r_{K,G} \) in \( \mathcal{S}_{n+2}(\Omega, \text{curl}, \text{div}0) \cap H^2(\Omega) \), we look for a minimizer \( \mathbf{v}_K \) of \( \mathcal{I}^r_{K,G} \) in \( \mathbb{I} \). From (4.1), Lemma 2.1 and the div-curl-gradient inequalities we see that

\[
\mathbb{I} \subset \mathcal{S}_{n+2}(\Omega, \text{curl}, \text{div}0, g) \subset H^1(\Omega, \mathbb{R}^3).
\]

Hence the functional \( \mathcal{I}^r_{K,G} \) is well defined on \( \mathbb{I} \). We shall also show that there exists a constant \( \beta_K^z \) such that the minimizer \( \mathbf{v}_K \) satisfies the following equality:

\[
\int_{\Omega} \left( \left( \mathcal{S}_{K}(|\text{curl} \, \mathbf{v}_K|^2) \right) \text{curl} \, \mathbf{v}_K \cdot \text{curl} \, \mathbf{w} - \frac{\beta_K^z}{2} \nabla_2 F(x, \mathbf{v}_K + \nabla \phi_{\nu_{K}}) \cdot \mathbf{w} \right) \, dx \\
+ \beta_K^z \int_{\partial \Omega} \left( D_T^0 \times w_T \right) \cdot dS = 0, \quad \forall \mathbf{w} \in \mathbb{I}_0.
\]

**Proposition 4.4.** Let \( 0 < K < + \infty \). Assume that \( \Omega, D_T^0, g, F, p, r \) satisfy (4.1), and \( G \) satisfies (G). Then the functional \( \mathcal{I}^r_{K,G} \) has a minimizer \( \mathbf{v}_K \) on \( \mathbb{I} \), and \( \mathbf{v}_K \) satisfies (4.4) with

\[
0 < \beta_K^z = G^r \left( \mathcal{I}^r[\mathbf{v}_K, \phi_{\nu_{K}}] - 2 \int_{\partial \Omega} (D_T^0 \times \mathbf{v}_{K,T}) \cdot dS \right) \leq M_2,
\]

where \( M_2 > 0 \) is the constant given in (G).

**Proof.** Step 1. For \( g \in H^{1/2}(\partial \Omega) \), let \( \mathbf{g}^\nu \in H^1(\Omega, \text{div}0, g) \) be a divergence-free extension of \( g \) in the sense that

\[
\text{div} \, \mathbf{g}^\nu = 0 \quad \text{in } \Omega, \quad \nu \cdot \mathbf{g}^\nu = g \quad \text{on } \partial \Omega.
\]

Moreover,

\[
\| \mathbf{g}^\nu \|_{H^1(\Omega)} \leq C(\Omega) \| g \|_{H^{1/2}(\partial \Omega)}.
\]
In fact, we can choose \( \mathbf{g}^e = \nabla \zeta \), where \( \zeta \in H^2(\Omega) \) is a weak solution of
\[
\Delta \zeta = 0 \quad \text{in } \Omega, \quad \frac{\partial \zeta}{\partial \nu} = g \quad \text{on } \partial \Omega.
\]
By the Sobolev embedding theorem we have \( \mathbf{g}^e \in L^p(\Omega, \mathbb{R}^3) \) for any \( 1 < p < 6 \). Thus \( \mathbf{g}^e \in \mathcal{S}^s_{\omega}(\Omega, \text{curl}, \text{div}0, g) \).

For any \( \mathbf{v} \in \mathcal{S}^s_{\omega}(\Omega, \text{curl}, \text{div}0, g) \) we can write \( \mathbf{v} = \mathbf{u} + \mathbf{g}^e \) with \( \mathbf{u} \in \mathcal{S}_0 \). If we define
\[
\mathcal{J}^e[\mathbf{u}] = \mathcal{H}_{K,G}[\mathbf{u} + \mathbf{g}^e + \nabla \phi_{u+g}^c],
\]
then
\[
\inf_{\mathbf{u} \in \mathcal{S}_0} \mathcal{J}^e[\mathbf{u}] = \inf_{\mathbf{v} \in \mathcal{S}} \mathcal{H}_{K,G}[\mathbf{v}].
\]
Using the condition on \( g \) in (4.1) and Lemma 2.1, for \( \mathbf{u} \in \mathcal{S}_0 \) we have \( \mathbf{u}_T \in TH^{1/2}(\partial \Omega, \mathbb{R}^3) \). By the condition on \( D_T^0 \) in (4.1) we see that the surface integral
\[
\int_{\partial \Omega} (D_T^0 \times \mathbf{u}_T) \cdot \nu \, dS
\]
is well-defined, and hence it defines a weakly continuous functional on \( \mathcal{S}_0 \). Combining this fact with Lemma 4.3 we see that the functional
\[
\mathbf{u} \mapsto G \left( \mathcal{F}^e[\mathbf{u} + \mathbf{g}^e, \phi_{u+g}^c] - 2 \int_{\partial \Omega} (D_T^0 \times (\mathbf{u}_T + \mathbf{g}_T^e)) \cdot \nu \, dS \right)
\]
is weakly continuous on \( \mathcal{S}_0 \). So the functional \( \mathcal{J}^e \) is weakly lower semi-continuous on \( \mathcal{S}_0 \). Similar to the proof of Proposition 3.4, we can show that \( \mathcal{J}^e \) is coercive in \( \mathcal{S}_0 \) with the equivalent norm shown in Lemma 2.3. Hence \( \mathcal{J}^e \) has a minimizer \( \mathbf{u}_K \in \mathcal{S}_0 \). Let
\[
\mathbf{v}_K = \mathbf{u}_K + \mathbf{g}^e.
\]
Then \( \mathbf{v}_K \) is a minimizer of \( \mathcal{H}_{K,G}^c \) on \( \mathcal{S} \).

Step 2. Similar to the proof of Lemma 3.5, we can prove that \( \mathbf{v}_K \) satisfies (4.4) with \( \beta_K^c \) given in (4.5). Here we omit the details. \( \square \)

Let \( \mathbf{A}_K = \mathbf{v}_K + \nabla \phi_{\mathbf{v}_K}^c \). Then \( \mathbf{A}_K \in \mathcal{S}^{2,p}(\Omega, \text{curl}) \). Now we show that \( \mathbf{A}_K \) is a weak solution of the system
\[
\begin{align*}
\text{curl} \left( S_K^c(\text{curl} \mathbf{A}^2) \text{curl} \mathbf{A} \right) - \frac{\beta_K^c}{2} \nabla z F(x, \mathbf{A}) &= \mathbf{U} \quad \text{in } \Omega, \\
S_K^c(\text{curl} \mathbf{A}^2)(\text{curl} \mathbf{A})_T &= \beta_K^c D_T^0 \quad \text{on } \partial \Omega,
\end{align*}
\]
for some \( \mathbf{U} \in \mathbb{H}_1(\Omega) \), with \( \beta_K^c \) a constant given in (4.5).

**Definition 4.5.** \( \mathbf{A} \in \mathcal{S}^{2,p}(\Omega, \text{curl}) \) is called a weak solution of (4.7) if the following equality holds for all \( \mathbf{H} \in \mathbb{H}^1(\Omega) \):
\[
\int_\Omega \left( S_K^c(\text{curl} \mathbf{A}^2) \text{curl} \mathbf{A} \cdot \text{curl} \mathbf{H} - \frac{\beta_K^c}{2} \nabla z F(x, \mathbf{A}) \cdot \mathbf{H} \right) \, dx + \beta_K^c \int_{\partial \Omega} (v \times D_T^0) \cdot \mathbf{H}_T \, dS = 0.
\]
Let us mention that, if \( \mathbf{A} \) satisfies (4.8) and if furthermore \( \mathbf{A} \in C^2(\bar{\Omega}, \mathbb{R}^3) \), then there exists a vector field \( \mathbf{U} \) with \( \mathbf{U} \in \mathbb{H}_1(\Omega) \) such that (4.7) holds.

**Proposition 4.6.** Let \( 0 < K < + \infty \). Assume that \( \Omega, D_T^0, g, F, p, r \) satisfy (4.1), and \( G \) satisfies (G). Let \( \mathbf{v}_K \) be the minimizer of \( \mathcal{H}_{K,G}^c \) obtained in Proposition 4.4. Then \( \mathbf{A}_K = \mathbf{v}_K + \nabla \phi_{\mathbf{v}_K}^c \) is a weak solution of (4.7). If furthermore \( \Omega \) is simply connected, then \( \mathbf{A}_K \) is a weak solution of (1.29) with \( \beta = \beta_K^c \).
Proof. Step 1. From Proposition 4.4 we know that \(v_K\) satisfies (4.4) for any \(w \in \mathcal{Z}_0\). Now we show that (4.8) holds for any \(H \in C^1(\bar{\Omega}, \mathbb{R}^3) \cap \mathbb{H}_+^1(\Omega)\). To prove, we decompose \(H\) as \(H = w + \nabla \psi\), where \(\psi \in W^{2,p}(\Omega)\) satisfies
\[
\Delta \psi = \text{div}H \quad \text{in} \quad \Omega, \quad \frac{\partial \psi}{\partial v} = v \cdot H \quad \text{on} \quad \partial \Omega,
\]
and \(w \in \mathcal{Z}_0\). Note that (4.4) holds for this \(w\). And the equality \(\text{curl}(\nabla \phi_{\varepsilon}^* ) = 0\) holds in the sense of \(L^2(\Omega, \mathbb{R}^3)\). Because \(\phi_{\varepsilon}^*\) is the minimizer of \(F^*[\cdot, \cdot]\) in \(W^{1,p}(\Omega)\), it satisfies (4.2) with \(v = v_K\) and with this \(\psi\) as a test function. Adding all these together we get (4.8) for all \(H \in C^1(\bar{\Omega}, \mathbb{R}^3) \cap \mathbb{H}_+^1(\Omega)\). The conclusion is followed since \(C^1(\bar{\Omega}, \mathbb{R}^3) \cap \mathbb{H}_+^1(\Omega)\) is dense in \(\mathbb{H}_+^1(\Omega)\).

Step 2. If \(\Omega\) is simply connected, then \(H_1(\Omega) = \{0\}\) and \(\mathbb{H}_+^1(\Omega) = L^2(\Omega, \mathbb{R}^3)\). Then (4.8) holds for all \(H \in C^1(\bar{\Omega}, \mathbb{R}^3)\). Thus \(A_K \in \mathfrak{S}_2^2(\Omega, \text{curl})\) is a weak solution of system (1.29).

Remark 4.7. In addition to the conditions of Proposition 4.6, if \(F\) and \(D^0\) satisfy the assumptions in Remark 4.2, then the weak solution \(A_K = v_K + \nabla \phi_{\varepsilon}^*\) of (4.7) satisfies an extra boundary condition
\[
\nu \cdot A = 0 \quad \text{on} \quad \partial \Omega.
\]

B. Estimates of the critical points of \(\mathcal{H}_{K,G}\)

In this subsection we only consider the special case where \(F(x, z)\) is given in (3.15) with \(a(x)\) satisfying (A2). Let \(0 < K < +\infty\). From Proposition 4.6, if \(\Omega\) is simply connected, then \(\mathcal{H}_{K,G}\) has a critical point \(A_K\) on \(\mathfrak{S}_2^{2,2}(\Omega, \text{curl})\), which is a weak solution of the Euler-Lagrange equation
\[
\begin{align*}
\text{curl} \left( S_K (|\text{curl} A|^2) \text{curl} A \right) - & \beta_K a(x) A = 0 \quad \text{in} \quad \Omega, \\
S_K (|\text{curl} A|^2)(\text{curl} A)_T = & \beta_K^* D_T^0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
and
\[
A_K = v_K + \nabla \phi_{\varepsilon}^* \in \mathfrak{S}_n^{2,2}(\Omega, \text{curl}, \text{div}0, g) + \text{grad} H^1(\Omega).
\]
Notice that \(g\) could be arbitrarily given, in the following for simplicity we set \(g = 0\), thus \(v_K \in \mathfrak{S}_n^{2,2}(\Omega, \text{curl}, \text{div}0)\).

We shall study the higher regularity of \(A_K\) and derive some estimates.

Proposition 4.8. Let \(0 < K < b^2\). Assume that \(\Omega, a(x), D^0_T\) satisfy (A1), (A2), and (A4), respectively, with \(0 < \alpha < 1\), and \(G\) satisfies (G). Let \(A_K \in \mathfrak{S}_n^{2,2}(\Omega, \text{curl})\) be a weak solution of (4.9) obtained in Proposition 4.6. Then
\[
A_K = v_K + \nabla \phi_{\varepsilon}^* \in C_0^{2,\alpha}(\bar{\Omega}, \text{div}0) + \text{grad} C_0^{2,\alpha}(\bar{\Omega}),
\]
and
\[
\|v_K\|_{C^{2,\alpha}()} + \|
abla \phi_{\varepsilon}^* \|_{C^{2,\alpha}()} \leq C,
\]
where \(C\) depends on \(\Omega, \alpha, a_0, \|a\|_{C^{2,\alpha}(\bar{\Omega})}, K, \delta, M_2, \|D_T^0\|_{C^{1,\alpha}(\partial \Omega)}, \text{and} \|\nu \cdot \text{curl} D_T^0\|_{C^{1,\alpha}(\partial \Omega)}\).

Proof. The proof of Proposition 4.8 is identical to the proof of Proposition 5.2 in Ref. 12 with \(\phi_K\) in Ref. 12 replaced by \(\phi_{\varepsilon}^*\), Eq. (5.7) of \(P_K\) in Ref. 12 replaced by
\[
\begin{align*}
\text{curl} P = & \beta_K^* a(x) A_K \quad \text{in} \quad \Omega, \\
\text{div} P = & 0 \quad \text{in} \quad \Omega, \\
P_T = & \beta_K^* D_T^0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
and Eq. (5.13) for \( \phi_K \) in Ref. 12 replaced by the following equation for \( \phi_{\nuK}^c \):

\[
\begin{align*}
\text{div}(a(x)\nabla\phi) &= -v_K \cdot \nabla a(x) \quad \text{in } \Omega, \\
a(x)\frac{\partial \phi}{\partial n} &= v \cdot \text{curl} D_T^0 \quad \text{on } \partial \Omega,
\end{align*}
\]

and accordingly the estimates are altered, and using the fact \( 0 < \beta_K^c \leq M_2 \).


Similar with Corollary 3.10 for Dirichlet problem, we can also derive higher regularity of the solution \( A_K \).

**Corollary 4.9.** In addition to the conditions of Proposition 4.8, assume furthermore that \( \partial \Omega \in C^{4, \alpha} \), and

\[ a(x) \in C^{2, \alpha}(\Omega), \quad D_T^0 \in C^{2, \alpha}(\partial \Omega, \mathbb{R}^3), \quad v \cdot \text{curl} D_T^0 \in C^{2, \alpha}(\partial \Omega). \]

Then

\[ A_K = v_K + \text{grad} \phi_{\nuK}^c \in C^{3, \alpha}(\bar{\Omega}, \text{div0}) + \text{grad} C^{3, \alpha}(\bar{\Omega}) \subset C^{2, \alpha}(\bar{\Omega}, \mathbb{R}^3), \]

\[ \|v_K\|_{C^{3, \alpha}(\bar{\Omega})} + \|\nabla \phi_{\nuK}^c\|_{C^{3, \alpha}(\bar{\Omega})} \leq C, \]

where \( C \) depends on \( \Omega, \alpha, a_0, \|a\|_{C^3(\bar{\Omega})}, K, \delta, M_2, \|D_T^0\|_{C^{2, \alpha}(\partial \Omega)} \) and \( \|v \cdot \text{curl} D_T^0\|_{C^{2, \alpha}(\partial \Omega)} \).

### C. Solutions of the Neumann problem (1.11)

In this subsection, we shall show the smallness of \( \|\text{curl} A_K\|_{L^\infty(\Omega)} \) for small boundary datum \( D_T^0 \), hence \( A_K \) is exactly the classical solution of Neumann problem (1.11). For this purpose we need the first eigenvalue of the following problem:

\[ \text{curl}^2 u = \mu u \quad \text{in } \Omega, \quad v \cdot u = 0 \quad \text{on } \partial \Omega. \]

We say that \( u \) is an eigenvector of (4.10) associated with an eigenvalue \( \mu \) if \( u \in \mathcal{H}^2(\Omega, \text{curl}), u \not\equiv 0 \) and it is a weak solution of (4.10).

**Lemma 4.10.** Assume that \( \Omega \) is a simply connected bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary. Then the first eigenvalue \( \mu_1 \) of (4.10) is positive and is given by

\[ \mu_1 = \inf_{u \in \mathcal{H}^2(\Omega, \text{curl}, \text{div0}), u \not\equiv 0} \frac{\|\text{curl} u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}. \]

**Proof.** We show \( \mu_1 > 0 \). Otherwise, there exists \( u \in \mathcal{H}^2(\Omega, \text{curl}, \text{div0}), u \not\equiv 0 \), such that \( \text{curl} u = 0 \). Then \( u \in \mathcal{H}^1(\Omega) \). But \( \mathcal{H}^1(\Omega) = \{0\} \) since \( \Omega \) is simply connected. Thus \( u = 0 \), a contradiction.

As mentioned before we can always choose the function \( G \) to be small. So we assume \( G \) satisfies (G) and

\[ 0 < M_2 < \frac{\mu_1 \min \mathcal{S}_K}{N_2(\Omega, a)}, \]

where \( N(\Omega, a) \) is given in (3.27).

**Proposition 4.11.** Let \( 0 < K < b^2 \). Assume that \( \Omega \) and \( a(x) \) satisfy (A1) and (A2), respectively, with \( 0 < \alpha < 1 \), and \( G \) is chosen to satisfy (G) and (4.11). Let

\[ A_K = v_K + \text{grad} \phi_{\nuK}^c \in C^{2, \alpha}(\bar{\Omega}, \text{div0}) + \text{grad} C^{2, \alpha}(\bar{\Omega}) \]

be the solution of system (4.9) obtained in Proposition 4.6. Then for any \( \varepsilon > 0 \) and \( \sigma \in [0, \alpha) \), there exists a constant \( \delta = \delta(\varepsilon, \sigma, M_2) > 0 \) such that if \( D_T^0 \) satisfies (A4) with

\[ \|D_T^0\|_{C^{1, \alpha}(\partial \Omega)} \leq \delta, \]

then \( \|A_K\|_{C^{1, \alpha}(\bar{\Omega})} \leq \varepsilon \).
Proof. The proof is similar to the proof of Proposition 3.12, with \( \lambda_1 \) replaced by \( \mu_1 \). Notice that \( \beta_K^c \) appears in the boundary condition for Neumann problem (4.9), so the constant \( \delta \) also depends on \( M_2 \) given in (G). \( \square \)

**Proof of Theorem 1.2.** Let us fix a constant \( K \) such that \( 0 < K < b^2 \). Then we find a function \( G \) satisfying (G) and (4.11). From Proposition 4.6, for any \( D_T^0 \) satisfying (A4), the system (4.9) with this \( S_K \) has a weak solution \( A_K = v_K + \nabla \phi_{v_K} \). From Proposition 4.8 we know that \( A_K \in C^{1,\alpha}(\Omega, \mathbb{R}^3) \). From Proposition 4.11 we know that there exists \( \delta > 0 \) such that if \( \|D_T^0\|_{C^{1,\alpha}(\Omega)} \leq \delta \), then \( \|\text{curl} A_K\|_{C(\overline{\Omega})} \leq \sqrt{K} \). Hence

\[
S_K(\|\text{curl} A_K(x)\|^2) = S'(\|\text{curl} A_K(x)\|^2), \quad \forall x \in \Omega.
\]

Therefore \( A_K \) is a solution of (1.11). Since \( K \) is fixed now, we denote \( A_K \) by \( A \) and denote \( \beta_K^c \) given in (4.5) by \( \beta \). Then \( A \) is a solution of (1.11) with the constant \( \beta \). \( \square \)

**D. Connection between the Dirichlet and Neumann problems**

In this subsection we show that the quasilinear Neumann problem

\[
\begin{align*}
\text{curl} \left( S'(\|\text{curl} A\|^2) \text{curl} A \right) + \epsilon a(x) A &= 0 \quad \text{in } \Omega, \\
S'(\|\text{curl} A\|^2)(\text{curl} A)_T &= D_T^0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(4.12)

where \( \epsilon = \pm 1 \), is equivalent to the semilinear Dirichlet problem

\[
\begin{align*}
\text{curl} \left( \frac{1}{a(x)} \text{curl} u \right) + \epsilon f(\|u\|^2) u &= 0 \quad \text{in } \Omega, \\
u_T &= D_T^0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(4.13)

where \( f(\rho) \) is the function given in Lemma 2.5, and the explicit meaning of “equivalence” will be clear below.

If \( A \in C^{2,\alpha}(\overline{\Omega}), \text{div} 0 \) + \( C^{2,\alpha}(\overline{\Omega}) \) is a solution of (4.12), we let \( B = \text{curl} A \) and taking curl on the both sides of (4.12) to obtain a new system

\[
\begin{align*}
\text{curl} \left[ \frac{1}{a(x)} \text{curl} \left( S'(\|B\|^2)B \right) \right] + \epsilon B &= 0 \quad \text{in } \Omega, \\
S'(\|B\|^2)B_T &= D_T^0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(4.14)

which can be written as (4.13), using the relation between \( B \) and \( u \) given in Lemma 2.5 (i).

On the other hand, if \( u \in C^{2,\alpha}(\overline{\Omega}), \text{div} 0 \) + \( C^{2,\alpha}(\overline{\Omega}) \) is a solution of (4.13) satisfying \( \|u\|_{C^{0}(\overline{\Omega})} < b/2 \), then we can find a solution \( A \) of (4.12) using this \( u \). In fact we first get \( B = f(\|u\|^2) u \) from this \( u \) using Lemma 2.5 (i), and hence \( B \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3) \) and satisfies (4.14). Now we look for a vector field \( A \) such that \( \text{curl} A = B \) and \( A \) solves (4.12). For this purpose we first look for a solution \( H \) of the following div-curl system:

\[
\begin{align*}
\text{curl} H &= B, \quad \text{div} H = 0 \quad \text{in } \Omega, \\
v \cdot H &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(4.15)

where the solvability of \( H \) is guaranteed since (4.13) implies that

\[
B = - \text{curl} \left( \frac{1}{a(x)} \text{curl} u \right) \in \text{curl} H^1(\Omega, \mathbb{R}^3).
\]

From (4.14) we know that

\[
\frac{1}{a(x)} \text{curl}(S'(\|B\|^2)B) + \epsilon H \in \text{Ker}(\text{curl}) = H^1(\Omega) \oplus \text{grad} H^1(\Omega),
\]

where \( v \) and \( H \) are to be determined.
and hence there exist \( U \in H^1_1(\Omega) \) and \( \phi \in H^1(\Omega) \) such that

\[
-\left[ \frac{1}{a(x)} \text{curl}(S'(\|B\|^2)B) + \epsilon \mathbf{H} \right] = U + \nabla \phi.
\]

Then \( A = \epsilon \mathbf{H} + U + \nabla \phi \) is a solution of (4.12).

Let us mention that the regularity of the solution \( H \) of (4.15) is determined by \( B \) and \( \Omega \), and the regularity of \( U \) is determined by \( \Omega \). From (4.12) we see that \( \text{div}(a(x)A) = 0 \), so \( \phi \) satisfies

\[
\begin{cases}
div(a(x)\nabla \phi) = -(\epsilon \mathbf{H} + U) \cdot \nabla a(x) & \text{in } \Omega, \\
a(x) \frac{\partial \phi}{\partial \nu} = -\nu \cdot \text{curl} D \text{ on } \partial \Omega,
\end{cases}
\]

and hence the regularity of \( \phi \) is determined by \( \Omega \), \( a(x) \), \( B \), \( U \), \( \nu \cdot \text{curl} D \).

By a similar discussion and using the relation between \( B \) and \( u \) given in Lemma 2.5 (ii), we see that the modified quasilinear equation

\[
\begin{cases}
\text{curl} \left( S'(\|B\|^2) \text{curl} A \right) + \epsilon a(x)A = 0 & \text{in } \Omega, \\
S'(\|B\|^2)(\text{curl} A) = D^0_T \text{ on } \partial \Omega,
\end{cases}
\]

is equivalent to the following semilinear equation:

\[
\begin{cases}
\text{curl} \left( \frac{1}{a(x)} \text{curl} u \right) + \epsilon f_K(\|u\|^2)u = 0 & \text{in } \Omega, \\
u_T = D^0_T \text{ on } \partial \Omega.
\end{cases}
\]

We believe that the methods in Subsections 4.1 and 4.2 of Ref. 12 can be used to obtain the existence and regularity of the solutions of (4.17).

V. THE THIRD TYPE PROBLEM

A. Proof of Theorem 1.3

1. Problem (1.14)

Step 1. We start with the modified equation

\[
\begin{cases}
\text{curl}(S'(|\text{curl} A|^2) \text{curl} A) + a(x)A = 0 & \text{in } \Omega, \\
\text{curl} A = B^0_T & \text{on } \partial \Omega.
\end{cases}
\]

which can be rewritten as the following equation for \( B = \text{curl} A \):

\[
\begin{cases}
\text{curl} \left[ \frac{1}{a(x)} \text{curl} \left( S'(|B|^2)B \right) \right] + B = 0 & \text{in } \Omega, \\
\text{div} B = 0 & \text{in } \Omega, \\
B_T = B^0_T & \text{on } \partial \Omega.
\end{cases}
\]

Since \( \Omega \) is a simply connected bounded domain in \( \mathbb{R}^3 \) without holes, and with a \( C^3,\alpha \) boundary, and \( B^0_T \in C^{2,\alpha}(\partial \Omega, \mathbb{R}^3) \), we can find a divergence-free extension \( B^\epsilon \) of \( B^0_T \) and \( B^\epsilon \in C^{2,\alpha}(\Omega, \mathbb{R}^3) \), see Refs. 8 and 18.

Step 2. Let us write \( B = B^\epsilon + w \) in (5.1) and define a map

\[
\begin{align*}
\mathcal{N}_K : C^{2,\alpha}(\Omega, \text{div}0) \times C^{2,\alpha}_0(\Omega, \text{div}0) & \to C^\alpha(\Omega, \text{div}0), \\
\mathcal{N}_K[B^\epsilon, w] & = \mathcal{M}_K[B^\epsilon + w], \\
\mathcal{M}_K[B] & = \text{curl} \left[ \frac{1}{a(x)} \text{curl} \left( S'(|B|^2)B \right) \right] + B.
\end{align*}
\]
Note that $M_K$ is a continuously differentiable map, with the Frechét derivative

$$M_K'[B](u) = \text{curl} \left( \frac{1}{a(x)} \text{curl} G_K(B, u) \right) + u,$$

where

$$G_K(B, u) = S'_K(|B|^2)u + 2S'_K(|B|^2)(B \cdot u)B.$$

Hence

$$\partial_w N_K[B^*, w] = M_K'[B]$$

is a bounded linear map from $C^{2, \alpha}_{0} (\tilde{\Omega}, \text{div}0)$ to $C^{\alpha} (\tilde{\Omega}, \text{div}0)$. Note that

$$G_K(0, u) = S'_K(0)u = S(0)u = \frac{1}{2} u.$$

For a given $f \in C^{\alpha}(\tilde{\Omega}, \text{div}0)$, the vector field $u \in C^{2, \alpha}_{0} (\tilde{\Omega}, \text{div}0)$ satisfies the equation $M_K'[0]u = f$ if and only if $u$ is a solution of the linear problem

$$\begin{aligned}
\frac{1}{2} \text{curl} \left( \frac{1}{a(x)} \text{curl} u \right) + u &= f \quad \text{in } \Omega, \\
\text{div} u &= 0 \quad \text{in } \Omega, \\
u_T &= 0 \quad \text{on } \partial \Omega.
\end{aligned}$$

Using the conditions (A1) and (A2') we know that the map

$$\mathcal{L}^+ u = \frac{1}{2} \text{curl} \left( \frac{1}{a(x)} \text{curl} u \right) + u$$

is a homeomorphism from $C^{2, \alpha}_{0} (\tilde{\Omega}, \text{div}0)$ onto $C^{\alpha}(\tilde{\Omega}, \text{div}0)$. Since

$$N_K[0, 0] = 0, \quad \partial_w N_K[0, 0] = M_K'[0] = \mathcal{L}^+, $$

hence $M_K'[0]$ has a bounded inverse.

So we can apply the implicit function theorem to conclude that there exist positive numbers $r_1$, $r_2$, with the sets

$$\mathfrak{B} = \{ B^* \in C^{2, \alpha}_{0}(\tilde{\Omega}, \text{div}0) : \| B^* \|_{C^{2, \alpha}(\tilde{\Omega})} < r_1 \},$$

$$\mathfrak{W} = \{ w \in C^{2, \alpha}_{0}(\tilde{\Omega}, \text{div}0) : \| w \|_{C^{2, \alpha}(\tilde{\Omega})} < r_2 \},$$

such that for any $B^* \in \mathfrak{B}$, there exists $w \in \mathfrak{W}$ such that

$$N_K[B^*, w] = 0.$$

and this $w$ is the unique solution in $\mathfrak{W}$. Moreover the map $B^* \mapsto w = W[B^*]$ is a continuously differentiable map from $\mathfrak{B}$ to $\mathfrak{W}$, and $W[0] = 0$. By the continuity of the map $W$ we can deduce $r_1$ hence deduce $r_2$ such that $r_1 + r_2 \leq \sqrt{K}$. Then we use the $C^{2, \alpha}$ estimate of the divergence-free extension (see Ref. 8) to find a constant $r_0 > 0$ with

$$\mathfrak{U} = \{ B^*_0 \in TC^{2, \alpha}(\partial \Omega, \mathbb{R}^3) : \| B^*_0 \|_{C^{2, \alpha}(\partial \Omega)} < r_0 \},$$

such that for any $B^*_0 \in \mathfrak{U}$ the divergence-free extension $B^* \in \mathfrak{B}$. Then $B = B^* + W[B^*]$ is a solution of (5.1). Since $\| B \|_{C^{2, \alpha}(\tilde{\Omega})} < r_1 + r_2 \leq \sqrt{K}$, $B$ is a solution of (5.1) with $S_K$ replaced by $S$.

Step 3. Let $B \in C^{2, \alpha}(\Omega, \text{div}0)$ be a solution of (5.1) with $S_K$ replaced by $S$. We look for a vector field $A$ such that $\text{curl} A = B$ and $A$ is a solution of (1.14). Since $\Omega$ is simply connected and has no holes, and $\text{div} B = 0$, the div-curl system (4.15) has a unique solution $H$ for this $B$. Since $\partial \Omega$ is of class $C^4, \alpha$ and $B \in C^{2, \alpha}(\bar{\Omega}, \text{div}0)$, hence $H \in C^{5, \alpha}(\bar{\Omega}, \text{div}0)$. Using (5.1) with $S_K$ replaced by $S$, we see that

$$\text{curl} \left[ \frac{1}{a(x)} \text{curl} (S'(|B|^2)B) + H \right] = 0.$$
Since \( \Omega \) is simply connected, there exists \( \phi \in C^{2,\alpha}(\bar{\Omega}) \) such that
\[
-\frac{1}{a(x)} \text{curl}(S(|B|^2)B) + H = \nabla \phi.
\]

Then \( A = H + \nabla \phi \) has the regularity shown in (1.17), and it is a solution of (1.14). Conclusion (i) is proved.

2. Problem (1.15)

By the same spirit we consider the following equation corresponding to the modified functional with a concave lower order term
\[
\begin{cases}
\text{curl}(S'_K(|\text{curl} A|^2) \text{curl} A) - a(x) A = 0 & \text{in } \Omega, \\
\text{(curl} A)_T = B_0^0 & \text{on } \partial \Omega.
\end{cases}
\tag{5.2}
\]

Most of analysis in the above works still with necessary modification. For instance, instead of the operator \( L^+ \), now the linearization leads to an linear operator
\[
L^- u = \frac{1}{2} \text{curl} \left( \frac{1}{a(x)} \text{curl} u \right) - u.
\]

If \( a(x) \) satisfies the condition (A2') then \( L^- \) is a homeomorphism from \( C^{2,\alpha}(\bar{\Omega}, \text{div} 0) \) onto \( C^\alpha(\bar{\Omega}, \text{div} 0) \) if and only if 2 is not an eigenvalue of \( L^+ \), namely \( \lambda = 2 \) is not an eigenvalue of (1.16). So we get the conclusion (ii).

We mention that if \( \Omega, B_0^0 \) and \( a \) has higher regularity then the solution \( A \) of (1.14) obtained in Theorem 1.3 has higher regularity.

Corollary 5.1. In addition to the conditions of Theorem 1.3, assume \( B_0^0 \in TC^{3,\alpha}(\partial\Omega, \mathbb{R}^3) \). Then the solutions of (1.14) and (1.15) obtained in Theorem 1.3 belong to \( C^{2,\alpha}(\bar{\Omega}, \mathbb{R}^3) \).

Proof. We give the proof for Eq. (1.14). We keep the notation in the proof of Theorem 1.3. Under the conditions of this corollary, the solution \( B \) of (5.1) belongs to \( C^{3,\alpha}(\bar{\Omega}, \mathbb{R}^3) \). The solution \( H \) of (4.15) belongs to \( C^{3,\alpha}(\Omega, \mathbb{R}^3) \) (which belongs to \( C^{4,\alpha}(\Omega, \text{div} 0) \) if \( \partial \Omega \) is of class \( C^{5,\alpha} \)). From (1.14) we furthermore see that \( \phi \) satisfies
\[
\begin{cases}
\text{div}(a(x)\nabla \phi) = -\nabla a(x) \cdot H & \text{in } \Omega, \\
a(x) \frac{\partial \phi}{\partial n} = -\nu \cdot \text{curl}(S'(|B|^2)B) & \text{on } \partial \Omega.
\end{cases}
\]

Now we have
\[
a(x) \in C^{2,\alpha}(\bar{\Omega}), \quad \nabla a(x) \cdot H \in C^{1,\alpha}(\bar{\Omega}, \text{div} 0), \quad \nu \cdot \text{curl}(S'(|B|^2)B) \in C^{2,\alpha}(\partial \Omega).
\]

From the regularity of higher order derivatives of linear Neumann problems we see that \( \phi \in C^{3,\alpha}(\bar{\Omega}) \). Thus \( A \in C^{3,\alpha}(\bar{\Omega}, \text{div} 0) + \text{grad } C^{3,\alpha}(\bar{\Omega}) \subset C^{2,\alpha}(\bar{\Omega}, \mathbb{R}^3) \).

B. Gauge invariant problem: The third type boundary condition

We say a problem is gauge invariant if \( A \) is a solution then \( A + \nabla \phi \) is also a solution, where \( \phi \) has certain regularity, see Ref. 12, Sec. VI. In this subsection we examine the gauge invariant equation with the third type boundary condition:
\[
\begin{cases}
\text{curl}(S'(|\text{curl} A|^2) \text{curl} A) + j = 0 & \text{in } \Omega, \\
\text{(curl} A)_T = B_0^0 & \text{on } \partial \Omega.
\end{cases}
\tag{5.3}
\]

Here we only require \( B_0^0 \in TC^{1,\alpha}(\partial \Omega, \mathbb{R}^3) \), and \( j \) satisfies the following condition:
(A6) \( j \in C^\alpha(\bar{\Omega}, \text{div} 0) \) with \( 0 < \alpha < 1 \).
We shall show that this system has infinitely many solutions, among them there is a solution satisfying extra conditions
\[
\text{div} A = 0 \quad \text{in } \Omega, \quad \nu \cdot A = 0 \quad \text{on } \partial \Omega.
\]

**Proposition 5.2.** Assume that \( \Omega \) and \( j \) satisfy (A1) and (A6), respectively, with \( 0 < \alpha < 1 \). Then there exists \( R_4 > 0 \) such that if
\[
B_{0,T} \in Tc^{1,\alpha}(\partial \Omega, \mathbb{R}^3), \quad \| B^0_T \|_{C^{1,\alpha}(\partial \Omega)} < R_4,
\]
the problem (5.3) has a solution \( A \in C^{2,\alpha}_{\text{loc}}(\Omega, \text{div}0) \). For any \( \phi \in H^1(\Omega) \), \( A + \nabla \phi \) is also a weak solution of (5.3).

**Proof. Step 1.** We first consider a solution \( A \) of the modification of (5.3) with \( S \) replaced by \( S_K \).

Then \( B = \text{curl} A \) is a solution of
\[
\begin{align*}
\text{curl} (S'_K(|B|^2)B) + j &= 0 \quad \text{in } \Omega, \\
\text{div} B &= 0 \quad \text{in } \Omega, \\
B_T &= B^0_T \quad \text{on } \partial \Omega. 
\end{align*}
\]  
(5.4)

Since \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) without holes and with a \( C^{2,\alpha} \) boundary and \( B^0_T \in C^{1,\alpha}(\partial \Omega, \mathbb{R}^3) \), we can find a divergence-free extension \( B^e \in C^{1,\alpha}(\bar{\Omega}, \mathbb{R}^3) \) of \( B^0_T \), and
\[
\| B^e \|_{C^{1,\alpha}(\bar{\Omega})} \leq C(\Omega, \alpha) \| B^0_T \|_{C^{1,\alpha}(\partial \Omega)},
\]  
(5.5)

see, for instance, Ref. 18. Write \( B = B^e + w \). Then (5.4) is reduced to
\[
\begin{align*}
\text{curl} [S'_K(|B^e + w|^2)(B^e + w)] + j &= 0 \quad \text{in } \Omega, \\
\text{div} w &= 0 \quad \text{in } \Omega, \\
w_T &= 0 \quad \text{on } \partial \Omega. 
\end{align*}
\]  
(5.6)

We define a map
\[
T_K : C^{1,\alpha}(\bar{\Omega}, \text{div}0) \times C^{1,\alpha}_{\text{loc}}(\bar{\Omega}, \text{div}0) \to C^\alpha(\bar{\Omega}, \text{div}0),
\]
\[
T_K[B^e, w] = R_K[B^e + w],
\]
\[
R_K[B] = \text{curl}(S'_K(|B|^2)B).
\]

Note that \( R_K \) is a continuously differentiable map, with the Frechét derivative
\[
R'_K[B](u) = \text{curl}(G_K(B, u)),
\]
where
\[
G_K(B, u) = S'_K(|B|^2)u + 2S''_K(|B|^2)(B \cdot u)B,
\]
and \( \partial_u T_K[B^e, w] = R'_K[B] \) is a bounded linear map from \( C^{1,\alpha}_{\text{loc}}(\bar{\Omega}, \text{div}0) \) to \( C^\alpha(\bar{\Omega}, \text{div}0) \).

We show that \( R'_K[0] \) has a bounded inverse. Given \( f \in C^\alpha(\bar{\Omega}, \text{div}0) \), the vector field \( u \in C^{2,\alpha}_c(\bar{\Omega}, \text{div}0) \) satisfies the equation \( R'_K[0]u = f \) if and only if \( u \) is a solution of the linear problem
\[
\text{curl } u = 2f, \quad \text{div } u = 0 \quad \text{in } \Omega, \quad u_T = 0 \quad \text{on } \partial \Omega.
\]

Since \( \Omega \) is simply connected and without holes, the above system has a unique solution \( u \) and
\[
\| u \|_{C^{1,\alpha}(\Omega)} \leq C(\Omega, \alpha) \| f \|_{C^\alpha(\Omega)}.
\]

Thus \( R'_K[0] \) is a homeomorphism from \( C^{1,\alpha}_{\text{loc}}(\bar{\Omega}, \text{div}0) \) to \( C^\alpha(\bar{\Omega}, \text{div}0) \).

**Step 2.** Since \( T_K[0, 0] = 0 \) and \( \partial_u T_K[0, 0] = R'_K[0] \), we can apply the implicit function theorem in the way similar to the first part of the proof of Theorem 1.3, and conclude that (5.4) has a solution \( B = B^e + \mathcal{W}_1[B^e] \), where \( \mathcal{W}_1[B^e] \) is a solution of
\[
T_K[B^e, w] = 0.
\]
Moreover, \(\|B\|_{C^{1,\alpha}(\Omega)} \leq \sqrt{K}\) if \(\|B\|_{C^{1,\alpha}(\partial \Omega)}\) is small. So \(B\) is a solution of (5.4) with \(S_K\) replaced by \(S\).

**Step 3.** Let \(B \in C^{1,\alpha}(\Omega, \text{div} 0)\) be a solution of (5.4) with \(S_K\) replaced by \(S\). Since \(\partial \Omega\) is simply-connected and has no holes, and \(\text{div} B = 0\), (4.15) has a unique solution \(H\). Since \(\partial \Omega\) is of class \(C^3\), \(B \in C^{1,\alpha}(\Omega, \text{div} 0)\), hence \(H \in C^{2,\alpha}(\Omega, \text{div} 0)\). Then \(H\) is a solution of (5.3). For any \(\phi \in H^1(\Omega)\), \(A = H + \nabla \phi\) is also a weak solution of (5.3).

\[\Box\]

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