GLOBAL SOLUTIONS OF TWO DIMENSIONAL INCOMPRESSIBLE VISCOELASTIC FLUID WITH DISCONTINUOUS INITIAL DATA

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ABSTRACT. The global existence of weak solutions of the incompressible viscoelastic flows in two spatial dimensions has been a long standing open problem, and it is studied in this paper. We show the global existence if the initial deformation gradient is close to the identity matrix in $L^2 \cap L^\infty$, and the initial velocity is small in L^2 and bounded in L^p, for some $p > 2$. While the assumption on the initial deformation gradient is automatically satisfied for the classical Oldroyd-B model, the additional assumption on the initial velocity being bounded in L^p for some $p > 2$ may due to techniques we employed. The smallness assumption on the L^2 norm of the initial velocity is, however, natural for the global well-posedness . One of the key observations in the paper is that the velocity and the "effective viscous flux"$\mathcal{G}$ are sufficiently regular for positive time. The regularity of $\mathcal{G}$ leads to a new approach for the pointwise estimate for the deformation gradient without using L^∞ bounds on the velocity gradients in spatial variables.

1. INTRODUCTION

The flow of incompressible viscoelastic fluids can be described by the following equations which are equivalent to the classical Oldroyd-B model (see [4, 5, 12, 14, 15]):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = \operatorname{div}(\mathtt{F}\mathtt{F}^\top), \\ \partial_t \mathtt{F} + \mathbf{u}\cdot\nabla\mathtt{F} = \nabla\mathbf{u}\mathtt{F}, \\ \operatorname{div}\mathbf{u} = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u} \in \mathbb{R}^2$ denotes the velocity of the fluid, $\mathtt{F} \in \mathcal{M}$ is the deformation gradient ($\mathcal{M}$ is the set of 2×2 matricies with $\det \mathtt{F} = 1$), and P is the pressure of the fluid, which is a Lagrangian multiplier due to the incompressibility of the fluid $\operatorname{div}\mathbf{u} = 0$. The viscosity μ is a positive constant, and will be assumed to be one throughout this paper for coneniences. If solutions $(\mathbf{u}, \mathtt{F})$ to (1.1) are smooth, it was well-known facts, see [1, 12, 14] that

$$\begin{cases} \operatorname{div}(\mathtt{F}^\top)(t) = 0, \\ \det \mathtt{F}(t) = 1 \end{cases} \tag{1.2}$$

for all $t > 0$ whenever (1.2) holds initially. Beside conserved quantities described in (1.2), it was also shown in [12, 19] that

$$\mathtt{F}_{lk}\partial_{x_l}\mathtt{F}_{ij}(t) = \mathtt{F}_{lj}\partial_{x_l}\mathtt{F}_{ik}(t) \tag{1.3}$$

for all $t > 0$ if (1.3) is valid initially.

Date: December 21, 2013.

2000 *Mathematics Subject Classification.* 35A05, 76A10, 76D03.

Key words and phrases. Incompressible viscoelastic fluid, weak solution, global well-posedness.

We consider here the Cauchy problem for the system (1.1), and the initial data will be specified by

$$(\mathbf{u}, \mathtt{F})|_{t=0} = (\mathbf{u}_0, \mathtt{F}_0)(x) \text{ for all } x \in \mathbb{R}^2. \tag{1.4}$$

One can easily generalize discussions here to the case of the initial-boundary value problem ([16]) or the Cauchy problem on a period box. For classical solutions of (1.1)-(1.4) or related Oldroyd-B models, authors in [1, 12, 14, 15, 5] have established various global existence and well-posedness of solutions to (1.1)-(1.4), say in H^2, whenever the initial data is a H^2 small perturbation around the equilibrium $(0, I)$, where I is the identity matrix. We refer to readers also [10, 19, 20, 21, 17, 18, 5] and references therein for local and global existence of solutions of closely related models. We shall point out in particular the works [20, 21], in which the authors used the hyperbolic nature of the system (1.1)-(1.4) when $\mu = 0$ to establish an interesting global existence result for classical solution in a subspace of H^s $(s \geq 8)$ when the initial date is also a small perturbation in that space of $(0, I)$ provided the spatial dimension is 3 due to the dispersive structure (see [13] for an almost global existence in dimension two). Note such a result is unknown for the Euler equations (when the elastic effects are not present). For the global existence of strong solutions near the equilibrium for compressible models of (1.1), we refer interested readers to [10, 19] and the references therein. Numerical evidence for singularities was provided in [23]. The regularity in terms of bounds on the elastic stress tensor was established in [2, 11]. In [6] authors proved global existence for small data with large gradients for Oldroyd-B. Regularity for diffusive Oldroyd-B equations in dimension two for large data were obtained in the creeping flow regime (coupling with the time independent Stokes equations, rather than Navier-Stokes) in [7] and in general in [8]. We note also that for the Oldroyd-B type models with a finite relaxation time the global existence of weak solutions with natural initial data had been verified in [17] under the corotational assumption. Recently a remarkable global existence result for weak solutions for the FENE dumbbell model with suitable initial data has been constructed by Masmoudi in [18] through a detailed analysis of the defect measure associated with the approximations. There is no such result for the Oldroyd-B model.The main result in this paper can be viewed therefore as the first step toward the solution of the corresponding problem for the Oldroyd-B model.

The construction of global solutions in [1, 12, 14, 15] depends crucially on various conserved quantities, in particular, (1.2) and (1.3) (see also [6]). Unfortunately, when the initial data are discontinuous, the proofs in [1, 6, 12, 14, 15] simply can not be made to work. On the other hand, One wishes to construct global solutions with the initial data in a natural functional space which can be read off from the basic energy law associated with sufficiently regular solutions of (1.1)-(1.4)

$$\frac{1}{2}\Big(\|\mathbf{u}\|_{L^2}^2 + \|\mathtt{F}\|_{L^2}^2\Big) + \mu \int_0^t \|\nabla \mathbf{u}(s)\|_{L^2}^2 ds = \frac{1}{2}\Big(\|\mathbf{u}_0\|_{L^2}^2 + \|\mathtt{F}_0\|_{L^2}^2\Big).$$

Thus for the classical Oldroyd-B model in two spatial dimensions, the natural initial velocity should be in L^2 and the deformation gradient can be chosen to be the identity (or small perturbations in $L^2 \cap L^\infty$ of I). To prove global existence of weak solutions under such initial conditions would therefore be of interest in theories of such fluids in both physics and mathematical analysis. From this point of view, this paper made a further step in understanding the system (1.1)-(1.4) with the constraints (1.2)-(1.3).

To facilitate the presentation, we introduce the notations

$$\varepsilon_0 = \|\mathtt{F}_0 - I\|_{L^\infty}^2 + \int_{\mathbb{R}^2} \left(|\mathtt{F}_0 - I|^2 + |\mathbf{u}_0|^2\right)(1+|x|^2)dx \tag{1.5}$$

where the weight $1+|x|^2$ serves to compensate the growth of the fundamental solution of the Laplacian at infinity (which is not needed when the spatial dimension is 3);

$$\begin{aligned} A(T) = \sup_{0\le t\le T}\int_{\mathbb{R}^2} \Big(|\mathbf{u}(x,t)|^2 + |\mathtt{F}(x,t)-I|^2 + \sigma(t)|\nabla\mathbf{u}(x,t)|^2 + \sigma(t)^2|\mathcal{P}\dot{\mathbf{u}}(x,t)|^2\Big)dx \\ + \int_0^T\int_{\mathbb{R}^2} \left(|\nabla\mathbf{u}|^2 + \sigma(t)|\dot{\mathbf{u}}|^2 + \sigma(t)^2|\nabla\mathcal{P}\dot{\mathbf{u}}|^2\right)dxdt, \end{aligned} \tag{1.6}$$

where $\dot{\mathbf{u}} = \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}$ is the material derivative of the velocity, $\sigma(t) = \min\{1,t\}$, and the operator $\mathcal{P}$ denotes the projection to the divergence free vector field; and

$$B(T) = \|\mathtt{F} - I\|_{L^\infty(\mathbb{R}^2\times[0,T])}^2. \tag{1.7}$$

The key difficulty to show the convergence of approximating solutions is to show the weak convergence of $\mathtt{FF}^\top$ at least in the sense of distributions, which requires a strong convergence of $\mathtt{F}$ in $L^2(\mathbb{R}^2)$. To overcome this difficulty, we introduce a quantity which is a suitable combination of effects from velocity and that of from the deformation gradient. This quantity will be called *effective viscous flux* , and it is defined as

$$\mathcal{G} = \nabla\mathbf{u} - (-\Delta)^{-1}\nabla\mathcal{P}\mathrm{div}(\mathtt{FF}^\top - I).$$

One can easily check from the first equation in (1.1) that

$$\Delta\mathcal{G} = \nabla\mathcal{P}\dot{\mathbf{u}}.$$

From this and (1.6), one can expect a bound of $\mathcal{G}$ in H^1 for positive time, which is better than either components of $\mathcal{G}$ that appeared to be.

We give a precise formulation of our results. First, denoting $\mathtt{F} = (\mathtt{F}_1, \mathtt{F}_2)$ where $\mathtt{F}_1$ and $\mathtt{F}_2$ are columns of $\mathtt{F}$, then the second equation in (1.1) can be written as

$$\partial_t\mathtt{F}_j + \mathbf{u}\cdot\nabla\mathtt{F}_j = \mathtt{F}_j\cdot\nabla\mathbf{u}$$

for $j=1,2$. Since $\mathrm{div}\mathtt{F}_j = 0$ due to $\mathrm{div}\mathtt{F}^\top = 0$, the equation for $\mathtt{F}_j$ can be further rewritten as

$$\partial_t\mathtt{F}_j + \mathrm{div}(\mathtt{F}_j\otimes\mathbf{u} - \mathbf{u}\otimes\mathtt{F}_j) = 0,$$

where $(a\otimes b)_{ij} = a_ib_j$. Next, we say that the pair $(\mathbf{u},\mathtt{F})$ is a weak solution of (1.1) with Cauchy data (1.4) provided that $\mathtt{F},\mathbf{u},\nabla\mathbf{u} \in L^1_{loc}(\mathbb{R}^2\times\mathbb{R}^+)$ and for all test functions $\beta,\psi\in\mathcal{D}(\mathbb{R}^2\times\mathbb{R}^+)$ with $\mathrm{div}\psi = 0$ in $\mathcal{D}'(\mathbb{R}^2\times\mathbb{R}^+)$

$$\int_{\mathbb{R}^2}(\mathtt{F}_j)_0\beta(\cdot,0)dx + \int_0^\infty\int_{\mathbb{R}^2}(\mathtt{F}_j\beta_t + (\mathtt{F}_j\otimes\mathbf{u} - \mathbf{u}\otimes\mathtt{F}_j):\nabla\beta)dxdt = 0 \tag{1.8}$$

for $j=1,2$, and

$$\begin{aligned} \int_{\mathbb{R}^2}\mathbf{u}_0\psi(\cdot,0)dx + \int_0^\infty\int_{\mathbb{R}^2}\Big[\mathbf{u}\cdot\partial_t\psi + (\mathbf{u}\otimes\mathbf{u} - \mathtt{FF}^\top):\nabla\psi\Big]dxdt \\ = \int_0^\infty\int_{\mathbb{R}^2}\nabla\mathbf{u}:\nabla\psi dxdt. \end{aligned} \tag{1.9}$$

Now, we are ready to state the main theorem.

Theorem 1.1. *Let $\|\mathbf{u}_0\|_{L^p} \le \alpha$ for some $p > 2$, and assume that $\varepsilon_0 \le \gamma$ for a sufficiently small γ that may depend on α and p. The Cauchy problem* (1.1)-(1.4) *with constraints* (1.2)-(1.3) *has a global weak solution* $(\mathbf{u}, \mathtt{F})$ *which is actually smooth for positive time. Moreover, there exist a positive constant θ that depends on p and a positive constant C that may depend on p and α such that*

$$A(t) \le C\varepsilon_0^\theta, \quad \textit{and} \quad B(t) \le C\varepsilon_0^\theta$$

for all $t \in \mathbb{R}^+$.

Remark 1.1. In [7], authors verified that the incompressible Navier-Stokes equations in dimension two forced by the divergence of a bounded stress have unique weak solutions, and in particular the weak solution is Holder continuous after an initial transient time. A new ingredient in our current work is to derive the L^∞ bound for the elastic stress via the trajectory. During the initial transient time, the Holder norms of the velocity will be compensated by the weight $\sigma(t)$.

Remark 1.2. We shall prove the above theorem under the assumption that $p = 4$ for saving some notations. It will be clear from the proofs presented below that the general case with $p > 2$ follows in the exactly the same manner. It should be also clear the similar proofs work also in dimension three. In the latter case, one needs to assume that the initial velocity to be small in L^3 and bounded in L^p for some $p > 3$. We note that the smallness of L^3 norm of the velocity is almost necessary even for the Navier-Stokes equations in dimension three. We believe that such a global existence theorem is also true when the initial data is small in a suitable Besov-space or a Lorentz space. For example, for the above theorem to be true in dimension two, one just need that the velocity is small in the Lorentz space $L^{2,1}$. But we do not prove the latter result in this paper partially because that it would make the article much more technical and longer. One may also conjecture that the above theorem is true when the velocity is small in L^2. The latter would require additional new ideas.

Theorem 1.1 will be established by passing to the limit as $n \to \infty$ of a sequence of approximating solutions $(\mathbf{u}^n, \mathtt{F}^n)$ which are global solutions of a modified system (1.1) with a biharmonic regularization for the velocity, $(-\Delta)^2\mathbf{u}$. This analysis then requires then us to derive a great deal of technical and qualitative information about the structure of these regularized flows. One is the need to find a mechanism in the solution operator which enforces appropriate pointwise bounds on the deformation gradient $\mathtt{F}$ in the absence of information concerning $\nabla\mathtt{F}$. More precisely, it is both physically significant and mathematically necessary to understand in detail precisely which quantities are smoothed out in the flow, which are not. One of the important observations we make here is that the gradient of the velocity, plus a suitable quantity relating the elastic stress, is in fact continuous.

The rest of this paper is organized as follows. In Section 2, we apply standard energy estimates to derive a bound for A (see Lemma 2.1). In Section 3, we introduce the *effective viscous flux* and estimate $A(t)$ in terms of ε_0, B (see Lemma 3.1). In Section 4, we derive the pointwise bounds for $\mathtt{F}$, and hence obtain a bound for B in terms of A (Lemma 3.1). Section 5 is devoted to the proof of Theorem 1.1. Throughout this paper, M will denote a generic positive constant which may depend on ε_0.

2. Basic Energy Estimate

In this section we derive certain *a priori* energy estimates for smooth solutions of (1.1). To begin with, let $(\mathbf{u}, \mathtt{F})$ be a smooth solution of (1.1) which is defined up to a positive time T and which satisfies the pointwise bounds

$$|\mathtt{F}(x,t) - I| \leq \frac{1}{2}.$$

Let ε_0 be as in (1.5), and we assume that ε_0, $A(T)$, $B(T) \leq 1$.

The bound of $A(T)$ can be stated as

Lemma 2.1.

$$A(T) \leq M\left(\varepsilon_0 + \int_0^T \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathbf{u}|^4 dxdt\right).$$

Proof. The proof consists of three separate energy-type estimates.

Step One: The first step is the energy-balance law. To derive it, we multiply the first equation and the second equation in (1.1) by $\mathbf{u}$ and $\mathtt{F}$ respectively, and then sum them together to obtain

$$\frac{1}{2}\partial_t(|\mathbf{u}|^2 + |\mathtt{F}|^2) + \frac{1}{2}\mathbf{u}\cdot\nabla(|\mathbf{u}|^2 + |\mathtt{F}|^2) - \sum_{i=1}^{2} \mathrm{div}(\nabla \mathbf{u}_i \mathbf{u}_i) + |\nabla \mathbf{u}|^2 + \mathrm{div}(P\mathbf{u}) = \partial_{x_j}(\mathtt{F}_{ik}\mathtt{F}_{jk}\mathbf{u}_i). \tag{2.1}$$

Here $|D| = \left(\sum_{i,j=1}^2 D_{ij}^2\right)^{\frac{1}{2}}$ for any 2×2 matrix D. On the other hand, we deduce from the second equation in (1.1) by taking the trace of the matrix

$$\partial_t \mathrm{tr}\mathtt{F} + \mathbf{u}\cdot\nabla \mathrm{tr}\mathtt{F} = \mathrm{tr}(\nabla \mathbf{u}\mathtt{F}). \tag{2.2}$$

Integrating (2.1) and (2.2) over $\mathbb{R}^2$ and using the facts $\mathrm{div}(\mathtt{F}^\top) = \mathrm{div}\mathbf{u} = 0$, one has

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|\mathbf{u}|^2 + |\mathtt{F} - I|^2)dx + \int_{\mathbb{R}^2}|\nabla \mathbf{u}|^2 dx = 0,$$

and hence

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^2}(|\mathbf{u}|^2 + |\mathtt{F} - I|^2)dx + 2\int_0^T\int_{\mathbb{R}^2}|\nabla \mathbf{u}|^2 dxdt = \int_{\mathbb{R}^2}(|\mathbf{u}_0|^2 + |\mathtt{F}_0 - I|^2)dx \leq \varepsilon_0. \tag{2.3}$$

Step Two: We derive estimates for the terms $\sigma\int_{\mathbb{R}^2}|\nabla \mathbf{u}|^2 dx$ and $\int_0^T\int_{\mathbb{R}^2}\sigma|\dot{\mathbf{u}}|^2 dxdt$ appearing in the definition of A. Applying the operator $\mathcal{P}$ to the first equation in (1.1), and taking L^2 inner product of the resulting equation with $\sigma\dot{\mathbf{u}}$, we obtain

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^2}\sigma|\dot{\mathbf{u}}|^2 dxds &= \int_0^t\int_{\mathbb{R}^2}\Big(\Delta \mathbf{u} + \mathcal{P}\mathrm{div}(\mathtt{F}\mathtt{F}^\top)\Big)\cdot\sigma\dot{\mathbf{u}}dxds \\ &\quad + \int_0^t\int_{\mathbb{R}^2}\mathcal{Q}(\mathbf{u}\cdot\nabla \mathbf{u})\cdot\sigma\dot{\mathbf{u}}dxds \\ &= I_1 + I_2 + I_3,\end{aligned} \tag{2.4}$$

where $\mathcal{Q} = Id - \mathcal{P}$. For I_1, we have

$$I_1 = \int_0^t\int_{\mathbb{R}^2}\sigma(\mathbf{u}_t + \mathbf{u}\cdot\nabla \mathbf{u})\cdot\Delta \mathbf{u}dxds = -\frac{\sigma}{2}\int_{\mathbb{R}^2}|\nabla \mathbf{u}|^2 dx + \frac{1}{2}\int_0^{\min\{1,t\}}\int_{\mathbb{R}^2}|\nabla \mathbf{u}|^2 dx + \mathcal{O}_3,$$

where $\mathcal{O}_3$ denotes a finite sum of terms of the form $\left|\int_0^t\int_{\mathbb{R}^2}\sigma\mathbf{u}^i_j\mathbf{u}^k_l\mathbf{u}^m_n dxds\right|$.

We can split I_2 as

$$\begin{aligned} I_2 &= \int_0^t\int_{\mathbb{R}^2}\mathrm{div}(\mathtt{F}\mathtt{F}^\top)\cdot\sigma\dot{\mathbf{u}}dxds - \int_0^t\int_{\mathbb{R}^2}\mathrm{div}(\mathtt{F}\mathtt{F}^\top)\cdot\sigma\mathcal{Q}(\mathbf{u}\cdot\nabla\mathbf{u})dxds\\ &= I_{2_1}+I_{2_2}. \end{aligned}$$

For I_{2_1}, we have, using $\mathrm{div}(\mathtt{F}^\top)=0$

$$\begin{aligned} I_{2_1} &= \int_0^t\int_{\mathbb{R}^2}\mathrm{div}(\mathtt{F}\mathtt{F}^\top)\cdot\sigma\dot{\mathbf{u}}dxds\\ &= \int_0^t\int_{\mathbb{R}^2}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\cdot(\sigma\dot{\mathbf{u}})dxds + \int_0^t\int_{\mathbb{R}^2}\mathrm{div}(\mathtt{F}-I)\cdot(\sigma\dot{\mathbf{u}})dxds\\ &= -\int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I)(\mathtt{F}-I)^\top:(\sigma\nabla\mathbf{u}_t+\sigma\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))dxds\\ &\quad -\int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I):(\sigma\nabla\mathbf{u}_t+\sigma\nabla(\mathbf{u}\cdot\nabla\mathbf{u})))dxds\\ &= -\int_{\mathbb{R}^2}\sigma(\mathtt{F}-I)(\mathtt{F}-I)^\top:\nabla\mathbf{u}dx + \int_0^t\int_{\mathbb{R}^2}\sigma_t(\mathtt{F}-I)(\mathtt{F}-I)^\top:\nabla\mathbf{u}dxds\\ &\quad +\int_0^t\int_{\mathbb{R}^2}\sigma\left((\mathtt{F}-I)(\mathtt{F}-I)^\top\right)_t:\nabla\mathbf{u}dxdt - \int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I)(\mathtt{F}-I)^\top:\sigma\nabla(\mathbf{u}\cdot\nabla\mathbf{u})dxds\\ &\quad -\int_{\mathbb{R}^2}\sigma(\mathtt{F}-I):\nabla\mathbf{u}dx + \int_0^t\int_{\mathbb{R}^2}\sigma_t(\mathtt{F}-I):\nabla\mathbf{u}dxds + \int_0^t\int_{\mathbb{R}^2}\sigma(\mathtt{F}-I)_t:\nabla\mathbf{u}dxds\\ &\quad -\int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I):\sigma\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))dxds. \end{aligned} \tag{2.5}$$

Note that for the last two terms in the most right hand side of (2.5), using the second equation of (1.1), we have, since $\mathrm{div}\mathbf{u}=0$

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^2}\sigma(\mathtt{F}-I)_t:\nabla\mathbf{u}dxds - \int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I):\sigma\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))dxds\\ &\quad = \int_0^t\int_{\mathbb{R}^2}\sigma(-\mathbf{u}\cdot\nabla(\mathtt{F}-I)+\nabla\mathbf{u}\mathtt{F}):\nabla\mathbf{u}dxds - \int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I):\sigma\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))dxds\\ &\quad = -\int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I):\sigma\nabla\mathbf{u}\nabla\mathbf{u}dxds + \int_0^t\int_{\mathbb{R}^2}\sigma\nabla\mathbf{u}\mathtt{F}:\nabla\mathbf{u}dxds\\ &\quad \le M\int_0^t\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dxds. \end{aligned}$$

Similarly, for the third term and the fourth term in the most right hand side of (2.5), we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^2}\sigma\left((\mathtt{F}-I)(\mathtt{F}-I)^\top\right)_t : \nabla\mathbf{u}dxdt-\int_0^t\int_{\mathbb{R}^2}(\mathtt{F}-I)(\mathtt{F}-I)^\top : \sigma\nabla(\mathbf{u}\cdot\nabla\mathbf{u})dxds\\
&=-\int_0^t\int_{\mathbb{R}^2}\sigma(\mathtt{F}-I)(\mathtt{F}-I)^\top : \nabla\mathbf{u}\nabla\mathbf{u}dxds\\
&\quad+\int_0^t\int_{\mathbb{R}^2}\sigma\left(\nabla\mathbf{u}\mathtt{F}(\mathtt{F}-I)^\top+(\mathtt{F}-I)(\nabla\mathbf{u}\mathtt{F})^\top\right) : \nabla\mathbf{u}dxds\\
&\le M\int_0^t\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dxds,
\end{aligned}$$

Hence, from those two estimates, we have

$$|I_{2_1}|\le M\Big(\sigma\int_{\mathbb{R}^2}|\nabla\mathbf{u}||\mathtt{F}-I|dx+\int_0^{\min\{1,t\}}\int_{\mathbb{R}^2}|\nabla\mathbf{u}||\mathtt{F}-I|dxds+\int_0^t\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dxds\Big).$$

For I_{2_2}, we have, since Riesz operator $\nabla\nabla(-\Delta)^{-1}$ is bounded in Hardy space $\mathcal{H}^1$

$$\begin{aligned}
I_{2_2}&=\int_0^t\int_{\mathbb{R}^2}\sigma\mathtt{F}\mathtt{F}^\top : \nabla\mathcal{Q}(\mathbf{u}\cdot\nabla\mathbf{u})dxds\\
&\le M\int_0^t\|\mathtt{F}\mathtt{F}^\top\|_{BMO}\|\nabla\mathcal{Q}(\mathbf{u}\cdot\nabla\mathbf{u})\|_{\mathcal{H}^1}ds\\
&\le M\int_0^t\|\mathtt{F}\mathtt{F}^\top\|_{L^\infty}\|\nabla\nabla(-\Delta)^{-1}(\partial_j\mathbf{u}^i\partial_i\mathbf{u}^j)\|_{\mathcal{H}^1}ds\\
&\le M\int_0^t\|\partial_j\mathbf{u}^i\partial_i\mathbf{u}^j\|_{\mathcal{H}^1}ds\\
&\le M\int_0^t\|\nabla\mathbf{u}\|_{L^2}^2ds.
\end{aligned}$$

Here, in the last inequality, we used the following estimate in [3]: if $\mathrm{div}v=0$, then

$$v\cdot\nabla w\in\mathcal{H}^1\quad\text{and}\quad\|v\cdot\nabla w\|_{\mathcal{H}^1}\le M\|v\|_{L^2}\|\nabla w\|_{L^2}.$$

For I_3, we have

$$\begin{aligned}
I_3&=\int_0^t\int_{\mathbb{R}^2}\nabla(-\Delta)^{-1}\mathrm{divdiv}(\mathbf{u}\otimes\mathbf{u})\cdot(\sigma\dot{\mathbf{u}})dxds\\
&=-\int_0^t\int_{\mathbb{R}^2}(-\Delta)^{-1}\mathrm{divdiv}(\mathbf{u}\otimes\mathbf{u})\cdot\sigma\mathrm{divdiv}(\mathbf{u}\otimes\mathbf{u})dxds\\
&\le M\int_0^t\sigma\|\mathrm{div}(\mathbf{u}\otimes\mathbf{u})\|_{L^2}^2ds \qquad (2.6)\\
&\le M\int_0^t\int_{\mathbb{R}^2}(|\mathbf{u}|^4+\sigma^2|\nabla\mathbf{u}|^4)dxds\\
&\le M\Big(\varepsilon_0+\int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4dxds\Big),
\end{aligned}$$

since for all $t \in [0,T]$,

$$\int_0^t \int_{\mathbb{R}^2} |\mathbf{u}(t)|^4 dxds \le \sup_{s\in[0,t]} \|\mathbf{u}(s)\|_{L^2}^2 \int_0^t \|\nabla\mathbf{u}(s)\|_{L^2}^2 ds \le \varepsilon_0^2 \le \varepsilon_0.$$

Summarizing those estimates for I_1, I_2 and I_3 together, we have

$$\begin{aligned}
&\frac{\sigma}{2}\int_{\mathbb{R}^2} |\nabla\mathbf{u}|^2 dx + \int_0^t\int_{\mathbb{R}^2} \sigma|\dot{\mathbf{u}}|^2 dxds \\
&\quad \le M\Big(\sigma\int_{R^2} |\nabla\mathbf{u}||\mathtt{F}-I|dx + \int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4 dxds + \varepsilon_0 \\
&\qquad + \int_0^{\min\{1,t\}}\int_{\mathbb{R}^2} |\nabla\mathbf{u}||\mathtt{F}-I|dxds + \int_0^t\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2 dxds + \mathcal{O}_3\Big).
\end{aligned}$$

This, together with (2.3) and Young's inequality, yields

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\sigma\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dx\right) + \int_0^T\int_{\mathbb{R}^2}\sigma|\dot{\mathbf{u}}|^2 dxdt \\
&\quad \le M\left(\varepsilon_0 + \int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4 dxds + \sum\left|\int_0^T\int_{\mathbb{R}^2}\sigma\mathbf{u}_j^i\mathbf{u}_l^k\mathbf{u}_n^m dxdt\right|\right).
\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}
\left|\int_0^T\int_{\mathbb{R}^2}\sigma\mathbf{u}_j^i\mathbf{u}_l^k\mathbf{u}_n^m dxds\right| &\le \int_0^T\int_{\mathbb{R}^2}\sigma|\nabla\mathbf{u}|^3 dxdt \\
&\le \left(\int_0^T\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2 dxdt\right)^{\frac12}\left(\int_0^T\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4 dxdt\right)^{\frac12} \\
&\le M\left(\varepsilon_0 + \int_0^T\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4 dxdt\right),
\end{aligned}$$

one deduces from (2.7) that

$$\sup_{0\le t\le T}\left(\sigma\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dx\right) + \int_0^T\int_{\mathbb{R}^2}\sigma|\dot{\mathbf{u}}|^2dxdt \le M\left(\varepsilon_0 + \int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4dxds\right). \tag{2.8}$$

Step Three: We estimate the terms $\sigma^2\int_{\mathbb{R}^2}|\mathcal{P}\dot{\mathbf{u}}|^2dx$ and $\int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathcal{P}\dot{\mathbf{u}}|^2dxds$ appearing in the definition of A. First of all, we apply the operator $\mathcal{P}$ to the first equation in (1.1) to yield

$$\mathcal{P}\dot{\mathbf{u}} - \Delta\mathbf{u} = \mathcal{P}\mathrm{div}(\mathtt{F}\mathtt{F}^\top).$$

Applying the operator $\partial_t + \mathbf{u}\cdot\nabla$ to the above equation gives

$$\partial_t(\mathcal{P}\dot{\mathbf{u}}) + \mathbf{u}\cdot\nabla\mathcal{P}\dot{\mathbf{u}} = \Delta\mathbf{u}_t + \mathrm{div}(\Delta\mathbf{u}\otimes\mathbf{u}) + (\mathcal{P}\mathrm{div}(\mathtt{F}\mathtt{F}^\top))_t + \mathrm{div}(\mathcal{P}\mathrm{div}(\mathtt{F}\mathtt{F}^\top)\otimes\mathbf{u}). \tag{2.9}$$

Multiplying (2.9) by $\sigma^2\mathcal{P}\dot{\mathbf{u}}$ and integrating over $\mathbb{R}^2\times(0,t)$, we obtain

$$\begin{aligned}\frac{1}{2}\sigma^2\int_{\mathbb{R}^2}|\mathcal{P}\dot{\mathbf{u}}|^2dx &= \int_0^t\int_{\mathbb{R}^2}\sigma\sigma'|\mathcal{P}\dot{\mathbf{u}}|^2dxds+\int_0^t\int_{\mathbb{R}^2}\sigma^2\mathcal{P}\dot{\mathbf{u}}\cdot\Big(\Delta\mathbf{u}_t+\mathrm{div}(\Delta\mathbf{u}\otimes\mathbf{u})\Big)dxds\\ &\quad+\int_0^t\int_{\mathbb{R}^2}\sigma^2\mathcal{P}\dot{\mathbf{u}}\cdot\Big((\mathcal{P}\mathrm{div}(\mathtt{FF}^\top))_t+\mathrm{div}(\mathcal{P}\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u})\Big)dxds\\ &=\sum_{j=1}^3 J_j.\end{aligned}\tag{2.10}$$

The estimate (2.8) can be used to control the first term on the right since $|\sigma'|\le 1$ and

$$\begin{aligned}|J_1| &= \left|\int_0^t\int_{\mathbb{R}^2}\sigma\sigma'|\mathcal{P}\dot{\mathbf{u}}|^2dxds\right|\\ &\le\int_0^t\int_{\mathbb{R}^2}\sigma|\dot{\mathbf{u}}|^2dxds.\end{aligned}$$

The second term J_2 on the right hand side (2.10) can be written as

$$\begin{aligned}J_2 &= \int_0^t\int_{\mathbb{R}^2}\sigma^2\mathcal{P}\dot{\mathbf{u}}\cdot\Big(\Delta\mathbf{u}_t+\mathrm{div}(\Delta\mathbf{u}\otimes\mathbf{u})\Big)dxds\\ &=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\left(\nabla\mathcal{P}\dot{\mathbf{u}}:\nabla\mathbf{u}_t+\nabla\mathcal{P}\dot{\mathbf{u}}:\Delta\mathbf{u}\otimes\dot{\mathbf{u}}\right)dxds\\ &=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\left(\nabla\mathcal{P}\dot{\mathbf{u}}:(\nabla\mathbf{u}_t+\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))+\nabla\mathcal{P}\dot{\mathbf{u}}:(\Delta\mathbf{u}\otimes\mathbf{u}-\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))\right)dxds.\end{aligned}\tag{2.11}$$

Note that

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:(\nabla\mathbf{u}_t+\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))dxds &= \int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\nabla\dot{\mathbf{u}}dxds\\ &=\int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathcal{P}\dot{\mathbf{u}}|^2dxds\end{aligned}$$

and integration by parts gives

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:(\Delta\mathbf{u}\otimes\mathbf{u}-\nabla(\mathbf{u}\cdot\nabla\mathbf{u}))dxds\\ &=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\Big((\mathbf{u}\cdot\nabla\partial_{x_l}\mathcal{P}\dot{\mathbf{u}})\cdot\partial_{x_l}\mathbf{u}+(\partial_{x_l}\mathbf{u}\cdot\nabla\mathcal{P}\dot{\mathbf{u}})\cdot\partial_{x_l}\mathbf{u}\Big)dxds\\ &\quad-\int_0^t\int_{\mathbb{R}^2}\sigma^2\Big(\partial_{x_l}\mathcal{P}\dot{\mathbf{u}}\cdot(\partial_{x_l}\mathbf{u}\cdot\nabla\mathbf{u})+\mathbf{u}\cdot\nabla\partial_{x_l}\mathbf{u}\cdot\partial_{x_l}\mathcal{P}\dot{\mathbf{u}}\Big)dxds\\ &=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\Big((\partial_{x_l}\mathbf{u}\cdot\nabla\mathcal{P}\dot{\mathbf{u}})\cdot\partial_{x_l}\mathbf{u}+\partial_{x_l}\mathcal{P}\dot{\mathbf{u}}\cdot(\partial_{x_l}\mathbf{u}\cdot\nabla\mathbf{u})\Big)dxds,\end{aligned}$$

since

$$\int_{\mathbb{R}^2}\Big((\mathbf{u}\cdot\nabla\partial_{x_l}\mathcal{P}\dot{\mathbf{u}})\cdot\partial_{x_l}\mathbf{u}+\mathbf{u}\cdot\nabla\partial_{x_l}\mathbf{u}\cdot\partial_{x_l}\mathcal{P}\dot{\mathbf{u}}\Big)dxds=0$$

due to $\mathrm{div}\mathbf{u}=0$. Therefore, we deduce from (2.11) that

$$J_2=-\int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathcal{P}\dot{\mathbf{u}}|^2dxds+\mathcal{O}_4,$$

where $\mathcal{O}_4$ denotes any term dominated by $M\int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^2|\nabla\mathcal{P}\dot{\mathbf{u}}|dxds$.

The third term J_3 on the right hand side of (2.10) can be written as

$$\begin{aligned}
J_3&=\int_0^t\int_{\mathbb{R}^2}\sigma^2\mathcal{P}\dot{\mathbf{u}}\cdot\Big((\mathcal{P}\mathrm{div}(\mathtt{FF}^\top))_t+\mathrm{div}(\mathcal{P}\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u})\Big)dxds\\
&=\int_0^t\int_{\mathbb{R}^2}\sigma^2\mathcal{P}\dot{\mathbf{u}}\cdot\Big(\mathcal{P}\mathrm{div}((\mathtt{FF}^\top)_t)+\mathrm{div}(\mathcal{P}\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u})\Big)dxds\\
&=\int_0^t\int_{\mathbb{R}^2}\sigma^2\mathcal{P}\dot{\mathbf{u}}\cdot\Big(\mathrm{div}((\mathtt{FF}^\top)_t)+\mathrm{div}(\mathcal{P}\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u})\Big)dxds\\
&=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\Big((\mathtt{FF}^\top)_t+\mathcal{P}\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u}\Big)dxds\\
&=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\Big((\mathtt{FF}^\top)_t+\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u}\Big)dxds\\
&\quad+\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\mathcal{Q}\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u}dxds\\
&=J_{3_1}+J_{3_2}.
\end{aligned}\tag{2.12}$$

¿From the second equation in (1.1), one has

$$\partial_t(\mathtt{FF}^\top)+\mathbf{u}\cdot\nabla(\mathtt{FF}^\top)=\nabla\mathbf{u}\mathtt{FF}^\top+\mathtt{FF}^\top(\nabla\mathbf{u})^\top,$$

Therefore, we can write J_{3_1} as, using integration by parts

$$\begin{aligned}
J_{3_1}&=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\Big((\mathtt{FF}^\top)_t+\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u}\Big)dxds\\
&=-\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\Big(-\mathbf{u}\cdot\nabla(\mathtt{FF}^\top)+\mathrm{div}(\mathtt{FF}^\top)\otimes\mathbf{u}\Big)dxds\\
&\quad-\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\Big(\nabla\mathbf{u}\mathtt{FF}^\top+\mathtt{FF}^\top(\nabla\mathbf{u})^\top\Big)dxds\\
&=\int_0^t\int_{\mathbb{R}^2}\sigma^2\partial_{x_jx_k}(\mathcal{P}\dot{\mathbf{u}})_i\Big(-\mathbf{u}_k(\mathtt{FF}^\top)_{ij}+(\mathtt{FF}^\top)_{ik}\mathbf{u}_j\Big)dxds\\
&\quad+\int_0^t\int_{\mathbb{R}^2}\sigma^2\partial_{x_j}(\mathcal{P}\dot{\mathbf{u}})_i(\mathtt{FF}^\top)_{ik}\partial_{x_k}\mathbf{u}_jdxds\\
&\quad-\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\Big(\nabla\mathbf{u}\mathtt{FF}^\top+\mathtt{FF}^\top(\nabla\mathbf{u})^\top\Big)dxds\\
&=\int_0^t\int_{\mathbb{R}^2}\sigma^2\partial_{x_j}(\mathcal{P}\dot{\mathbf{u}})_i(\mathtt{FF}^\top)_{ik}\partial_{x_k}\mathbf{u}_jdxds\\
&\quad-\int_0^t\int_{\mathbb{R}^2}\sigma^2\nabla\mathcal{P}\dot{\mathbf{u}}:\Big(\nabla\mathbf{u}\mathtt{FF}^\top+\mathtt{FF}^\top(\nabla\mathbf{u})^\top\Big)dxds
\end{aligned}$$

since interchanging j and k yields

$$\int_0^t \int_{\mathbb{R}^2} \sigma^2 \partial_{x_j x_k} (\mathcal{P}\dot{\mathbf{u}})_i \Big(-\mathbf{u}_k (\mathtt{FF}^\top)_{ij} + (\mathtt{FF}^\top)_{ik} \mathbf{u}_j \Big) dxds = 0.$$

Thus, it follows

$$\begin{aligned} |J_{3_1}| &\le M \left(\int_0^t \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dxds \right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathcal{P}\dot{\mathbf{u}}|^2 dxds \right)^{\frac{1}{2}} \\ &\le M \varepsilon_0^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathcal{P}\dot{\mathbf{u}}|^2 dxds \right)^{\frac{1}{2}}. \end{aligned} \tag{2.13}$$

On the other hand, one can write J_{3_2} as

$$\begin{aligned} J_{3_2} &= \int_0^t \int_{\mathbb{R}^2} \sigma^2 \nabla \mathcal{P}\dot{\mathbf{u}} : \mathcal{Q}\mathrm{div}(\mathtt{FF}^\top) \otimes \mathbf{u} dxds \\ &= \int_0^t \int_{\mathbb{R}^2} \sigma^2 \nabla \mathcal{P}\dot{\mathbf{u}} : \nabla(-\Delta)^{-1} \mathrm{divdiv}(\mathtt{FF}^\top) \otimes \mathbf{u} dxds \\ &= \int_0^t \int_{\mathbb{R}^2} \sigma^2 \partial_{x_j} (\mathcal{P}\dot{\mathbf{u}})_i \partial_{x_i} (-\Delta)^{-1} \mathrm{divdiv}(\mathtt{FF}^\top) \mathbf{u}_j dxds \\ &= -\int_0^t \int_{\mathbb{R}^2} \sigma^2 \partial_{x_j} \partial_{x_i} (\mathcal{P}\dot{\mathbf{u}})_i (-\Delta)^{-1} \mathrm{divdiv}(\mathtt{FF}^\top) \mathbf{u}_j dxds \\ &\quad - \int_0^t \int_{\mathbb{R}^2} \sigma^2 \partial_{x_j} (\mathcal{P}\dot{\mathbf{u}})_i (-\Delta)^{-1} \mathrm{divdiv}(\mathtt{FF}^\top) \partial_{x_i} \mathbf{u}_j dxds \\ &= -\int_0^t \int_{\mathbb{R}^2} \sigma^2 \partial_{x_j} (\mathcal{P}\dot{\mathbf{u}})_i (-\Delta)^{-1} \mathrm{divdiv}(\mathtt{FF}^\top) \partial_{x_i} \mathbf{u}_j dxds \end{aligned} \tag{2.14}$$

since $\partial_{x_i} (\mathcal{P}\dot{\mathbf{u}})_i = \mathrm{div}(\mathcal{P}\dot{\mathbf{u}}) = 0$. Therefore, we can estimate J_{3_2} as

$$\begin{aligned} |J_{3_2}| &\le \int_0^t \sigma^2 \|(-\Delta)^{-1} \mathrm{divdiv}(\mathtt{FF}^\top)\|_{BMO} \|\partial_{x_j} (\mathcal{P}\dot{\mathbf{u}})_i \partial_{x_i} \mathbf{u}_j\|_{\mathcal{H}^1} ds \\ &\le M \int_0^t \sigma^2 \||\mathtt{F}|^2\|_{L^\infty} \|\nabla \mathcal{P}\dot{\mathbf{u}}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} ds \\ &\le M \varepsilon_0^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathcal{P}\dot{\mathbf{u}}|^2 dxds \right)^{\frac{1}{2}}. \end{aligned}$$

Substituting (2.13) and (2.14) back to (2.12) gives

$$|J_3| \le M \varepsilon_0^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathcal{P}\dot{\mathbf{u}}|^2 dxds \right)^{\frac{1}{2}}.$$

Summarizing estimates for J_j (j=1,2,3) in (2.10) and using Young's inequality, one obtains

$$\sigma^2 \int_{\mathbb{R}^2} |\mathcal{P}\dot{\mathbf{u}}|^2 dx + \int_0^t \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathcal{P}\dot{\mathbf{u}}|^2 dxds \le M \left(\varepsilon_0 + \int_0^t \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathbf{u}|^4 dxds \right).$$

It then follows easily that

$$\sup_{0<t\le T}\sigma^2\int_{\mathbb{R}^2}|\mathcal{P}\dot{\mathbf{u}}|^2dx+\int_0^T\int_{\mathbb{R}^2}\sigma^2|\nabla\mathcal{P}\dot{\mathbf{u}}|^2dxds\le M\left(\varepsilon_0+\int_0^T\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4dxds\right).$$

$\square$

3. Effective viscous flux and Estimate for $A(t)$

Let us define *effective viscous flux* as

$$\mathcal{G}=\nabla\mathbf{u}-(-\Delta)^{-1}\nabla\mathcal{P}\mathrm{div}(\mathtt{F}\mathtt{F}^\top-I),$$

and its variant

$$\mathfrak{G}=\nabla\mathbf{u}+\mathtt{F}-I.$$

The condition $\mathrm{div}\mathtt{F}^\top=0$ implies that

$$\mathcal{P}\mathrm{div}(\mathtt{F}-I)=\mathrm{div}(\mathtt{F}-I),\tag{3.1}$$

and hence, in view of the identity

$$\Delta=\nabla\mathrm{div}-\mathrm{curlcurl},$$

one has

$$\begin{aligned}\Delta\mathfrak{G}&=\Delta\Big(\mathcal{G}+(-\Delta)^{-1}\nabla\mathcal{P}\mathrm{div}(\mathtt{F}\mathtt{F}^\top-I)+\mathtt{F}-I\Big)\\&=\Delta\Big(\mathcal{G}+(-\Delta)^{-1}\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)+(-\Delta)^{-1}\mathrm{curlcurl}(\mathtt{F}-I)\Big)\\&=\Delta\mathcal{G}-\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)-\mathrm{curlcurl}(\mathtt{F}-I).\end{aligned}\tag{3.2}$$

¿From the first equation in (1.1), we have

$$\Delta\mathbf{u}+\mathrm{div}(\mathtt{F}-I)=\mathcal{P}\dot{\mathbf{u}}-\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top),$$

and thus one has, using (3.1)

$$\begin{aligned}&(\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top),\Delta\mathfrak{G})\\&\quad=\Big(\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top),\nabla(\Delta\mathbf{u}+\mathrm{div}(\mathtt{F}-I))\Big)\\&\quad=\Big(\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top),\nabla\mathcal{P}\dot{\mathbf{u}}\Big)-\Big\|\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\Big\|_{L^2}^2\\&\quad\le-\frac12\Big\|\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\Big\|_{L^2}^2+\frac12\|\nabla\mathcal{P}\dot{\mathbf{u}}\|_{L^2}^2.\end{aligned}\tag{3.3}$$

On the other hand, it also holds

$$\begin{aligned}(\mathrm{curlcurl}(\mathtt{F}-I),\Delta\mathfrak{G})&=-(\mathrm{curlcurl}(\mathtt{F}-I),\mathrm{curlcurl}\mathfrak{G})\\&=-\|\mathrm{curlcurl}(\mathtt{F}-I)\|_{L^2}^2.\end{aligned}\tag{3.4}$$

¿From (3.2), (3.3), and (3.4), one deduces that

$$\begin{aligned}
&\|\Delta\mathfrak{G}\|_{L^2}^2+\|\mathrm{curlcurl}(\mathtt{F}-I)\|_{L^2}^2+\left\|\nabla\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\right\|_{L^2}^2\\
&\le M\Big(\|\Delta\mathcal{G}\|_{L^2}^2+\|\nabla\mathcal{P}\dot{\mathbf{u}}\|_{L^2}^2\Big)\\
&\le M\|\nabla\mathcal{P}\dot{\mathbf{u}}\|_{L^2}^2,
\end{aligned}\tag{3.5}$$

since $\Delta\mathcal{G}=\nabla\mathcal{P}\dot{\mathbf{u}}$. Similarly, it also holds

$$\|\nabla\mathfrak{G}\|_{L^2}\le M\|\mathcal{P}\dot{\mathbf{u}}\|_{L^2}^2.$$

Those inequalities imply that the quantity $\mathfrak{G}$ has the same regularity as the quantity $\mathcal{G}$.

We are now in a position to obtain the required bounds for the terms $\int_0^t\int_{\mathbb{R}^2}\sigma^2|\nabla\mathbf{u}|^4dxds$ appearing in the statement of Lemma 2.1

Lemma 3.1. *There is a global positive constant θ such that*

$$A(T)\le M\Big(\varepsilon_0^\theta+A(T)^2+B(T)^2\Big).$$

Proof. ¿From the definition of $\mathfrak{G}$ and the second equation in (1.1), we have that

$$\frac{d}{dt}(\mathtt{F}-I)+\mathtt{F}-I=\mathfrak{G}\mathtt{F}-(\mathtt{F}-I)(\mathtt{F}-I).$$

Multiplying by $4(\mathtt{F}-I)|\mathtt{F}-I|^2$, we obtain

$$\frac{d}{dt}|\mathtt{F}-I|^4+4|\mathtt{F}-I|^4\le 4|\mathfrak{G}||\mathtt{F}||\mathtt{F}-I|^3+4|\mathtt{F}-I|^5,$$

and hence, the bound $\|\mathtt{F}-I\|_{L^\infty}\le\frac12$ implies that

$$\frac{d}{dt}|\mathtt{F}-I|^4+|\mathtt{F}-I|^4\le M|\mathfrak{G}|^4.\tag{3.6}$$

Multiplying by σ^2 and integrating along the trajectory yields

$$\begin{aligned}
&\sigma^2(T)|\mathtt{F}(x(T),T)-I|^4+\int_0^T\sigma^2(t)|\mathtt{F}(x(t),t)-I|^4dt\\
&\quad\le M\int_0^{\min\{1,T\}}|\mathtt{F}(x(t),t)-I|^4dt+M\int_0^T\sigma^2(t)\mathfrak{G}^4dt.
\end{aligned}$$

Integrating over $\mathbb{R}^2$ and using the fact $\det\mathtt{F}=1$, one obtains

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}\sigma^2(t)|\mathtt{F}(x,t)-I|^4dxdt\\
&\le M\int_0^{\min\{1,T\}}\int_{\mathbb{R}^2}|\mathtt{F}(x,t)-I|^4dxdt+M\int_0^T\int_{\mathbb{R}^2}\sigma^2(t)\mathfrak{G}^4dxdt\\
&\le M\varepsilon_0B+M\int_0^T\sigma^2(t)\mathfrak{G}^4dt.
\end{aligned}$$

The definition of $\mathcal{G}$ implies

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathbf{u}|^4 dxdt \\
&\quad \le M\Big(\int_0^T \int_{\mathbb{R}^2} \sigma^2 |\mathcal{G}|^4 dxdt + \int_0^T \int_{\mathbb{R}^2} \sigma^2 |(-\Delta)^{-1} \nabla \mathcal{P} \mathrm{div}(\mathtt{F}\mathtt{F}^\top - I)|^4 dxdt \Big) \\
&\quad \le M\Big(\int_0^T \int_{\mathbb{R}^2} \sigma^2 |\mathcal{G}|^4 dxdt + \int_0^T \sigma^2 \|\mathtt{F}\mathtt{F}^\top - I\|_{L^4}^4 dt \Big) \\
&\quad \le M\Big(\int_0^T \int_{\mathbb{R}^2} \sigma^2 \Big[|\mathcal{G}|^4 + \mathfrak{G}|^4 \Big] dxdt + \varepsilon_0 B \Big).
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^2} \sigma^2 \mathcal{G}^4 dxdt &\le \int_0^T \sigma^2 \left(\int_{\mathbb{R}^2} \mathcal{G}^2 dx \right) \left(\int_{\mathbb{R}^2} |\nabla \mathcal{G}|^2 dx \right) dt \\
&\le \sup_t \left[\left(\sigma \int_{\mathbb{R}^2} \mathcal{G}^2 dx \right) \right] \int_0^T \int_{\mathbb{R}^2} \sigma |\nabla \mathcal{G}|^2 dxdt
\end{aligned}$$

However, from the definition of $\mathcal{G}$,

$$\begin{aligned}
\sigma \int_{\mathbb{R}^2} \mathcal{G}^2 dx &\le M \left[\int_{\mathbb{R}^2} |\mathtt{F} - I|^2 dx + \sigma \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx \right] \\
&\le M(\varepsilon_0 + A(T)).
\end{aligned}$$

Also, since $\Delta \mathcal{G} = \nabla \mathcal{P} \dot{\mathbf{u}}$,

$$\int_{\mathbb{R}^2} |\nabla \mathcal{G}|^2 dx \le M \int_{\mathbb{R}^2} |\mathcal{P} \dot{\mathbf{u}}|^2 dx.$$

Applying these bounds, we obtain that

$$\int_0^T \int_{\mathbb{R}^2} \sigma^2 \mathcal{G}^4 \le M(\varepsilon_0^2 + A(T)^2),$$

and similarly

$$\int_0^T \int_{\mathbb{R}^2} \sigma^2 \mathfrak{G}^4 \le M(\varepsilon_0^2 + A(T)^2).$$

Hence

$$\int_0^T \int_{\mathbb{R}^2} \sigma^2 |\nabla \mathbf{u}|^4 dxdt \le M(\varepsilon_0^2 + A(T)^2 + B(T)^2).$$

□

In the following lemma we derive an estimate for the weighted L^2-norm of $\mathbf{u}(\cdot, t)$ for $0 \le t \le 1$.

Lemma 3.2. *Under the same condition of Theorem 1.1, it holds*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^2} (1 + |x|^2) |\mathbf{u}(x,t)|^2 dx \le M(T) \varepsilon_0.$$

Proof. Taking the inner product of the first equation of (1.1) with $(1+|x|^2)\mathbf{u}$, one obtains

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}(1+|x|^2)|\mathbf{u}|^2dx\Big|_0^t+\int_0^t\int_{\mathbb{R}^2}(1+|x|^2)|\nabla\mathbf{u}|^2dxds\\
&\quad=-\int_0^t\int_{\mathbb{R}^2}\Big[x\cdot\nabla|\mathbf{u}|^2-\mathcal{Q}\mathrm{div}(\mathbf{u}\mathbf{u}_j)\cdot(1+|x|^2)\mathbf{u}-|\mathbf{u}|^2x\cdot\mathbf{u}+(1+|x|^2)\mathbf{u}\cdot\mathrm{div}(\mathtt{F}-I)\\
&\qquad+\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\cdot\mathbf{u}(1+|x|^2)\Big]dxds\\
&\quad=-\int_0^t\int_{\mathbb{R}^2}\Big[x\cdot\nabla|\mathbf{u}|^2+2(-\Delta)^{-1}\mathrm{divdiv}(\mathbf{u}\otimes\mathbf{u})x\cdot\mathbf{u}-|\mathbf{u}|^2x\cdot\mathbf{u}\\
&\qquad+(1+|x|^2)\mathbf{u}\cdot\mathrm{div}(\mathtt{F}-I)+\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\cdot\mathbf{u}(1+|x|^2)\Big]dxds.
\end{aligned}
$$

In a similar way, we find from the second equation of (1.1) that

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}(1+|x|^2)|\mathtt{F}-I|^2dx\Big|_0^t\\
&\quad=\int_0^t\int_{\mathbb{R}^2}\Big[|\mathtt{F}-I|^2\mathbf{u}\cdot x-\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\cdot\mathbf{u}(1+|x|^2)\\
&\qquad-(1+|x|^2)\mathbf{u}\cdot\mathrm{div}(\mathtt{F}-I)-2(\mathtt{F}-I)(\mathtt{F}-I):\mathbf{u}\otimes x-2(\mathtt{F}-I):\mathbf{u}\otimes x\Big]dxds.
\end{aligned}
$$

Adding them together, we thus obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}(1+|x|^2)\Big[|\mathbf{u}|^2+|\mathtt{F}-I|^2\Big]dx\Big|_0^t+\int_0^t\int_{\mathbb{R}^2}(1+|x|^2)|\nabla\mathbf{u}|^2dxds\\
&\quad=-\int_0^t\int_{\mathbb{R}^2}\Big[|\mathtt{F}-I|^2\mathbf{u}\cdot x+x\cdot\nabla|\mathbf{u}|^2-\mathcal{Q}\mathrm{div}(\mathbf{u}\mathbf{u}_j)\cdot(1+|x|^2)\mathbf{u}-|\mathbf{u}|^2x\cdot\mathbf{u}\\
&\qquad-2(\mathtt{F}-I)(\mathtt{F}-I):\mathbf{u}\otimes x-2(\mathtt{F}-I):\mathbf{u}\otimes x\\
&\qquad-\mathcal{Q}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\cdot\mathbf{u}(1+|x|^2)\Big]dxds\\
&\quad=-\int_0^t\int_{\mathbb{R}^2}\Big[|\mathtt{F}-I|^2\mathbf{u}\cdot x+x\cdot\nabla|\mathbf{u}|^2+2(-\Delta)^{-1}\mathrm{divdiv}(\mathbf{u}\otimes\mathbf{u})x\cdot\mathbf{u}-|\mathbf{u}|^2x\cdot\mathbf{u}\\
&\qquad-2(\mathtt{F}-I)(\mathtt{F}-I):\mathbf{u}\otimes x-2(\mathtt{F}-I):\mathbf{u}\otimes x\\
&\qquad-2(-\Delta)^{-1}\mathrm{divdiv}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\mathbf{u}\cdot x\Big]dxds.
\end{aligned}
\tag{3.7}
$$

The right hand side of (3.7) is controlled by

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}(1+|x|^2)|\nabla\mathbf{u}|^2dxds+M\int_0^t\int_{\mathbb{R}^2}(1+|x|^2)\Big[|\mathbf{u}|^2+|\mathtt{F}-I|^2\Big]dxds\\
&\quad+\int_0^t\int_{\mathbb{R}^2}|\mathbf{u}|^4dxds.
\end{aligned}
$$

The last term above is bounded by

$$
M\int_0^t\left(\int_{\mathbb{R}^2}|\mathbf{u}|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dx\right)^{\frac{1}{2}}dt\le M\varepsilon_0t^{\frac{1}{2}}.
$$

Thus, we deduce from (3.7) that

$$\begin{aligned}&\int_{\mathbb{R}^2}(1+|x|^2)(|\mathbf{u}|^2+|\mathtt{F}-I|^2)dx\\&\quad\le M\Big[\varepsilon_0 t^{\frac12}+\int_0^t\int_{\mathbb{R}^2}(1+|x|^2)(|\mathbf{u}|^2+|\mathtt{F}-I|^2)dxds\Big].\end{aligned}$$

An easy application of Gronwall's inequality then completes the proof. □

In the following lemma we derive an estimate for $\|\mathbf{u}(t)\|_{L^4}$ as $0\le t\le 1$.

Lemma 3.3. *Assume that* $\mathbf{u}_0\in L^4$. *Then*

$$\begin{aligned}&\sup_{0\le t\le T}\int_{\mathbb{R}^2}|\mathbf{u}|^4dx+\int_0^T\int_{\mathbb{R}^2}|\mathbf{u}|^2|\nabla\mathbf{u}|^2dxdt\\&\le M(T)\Big[\int_{\mathbb{R}^2}|\mathbf{u}_0|^4dx+\varepsilon_0 B\Big].\end{aligned}$$

Proof. Taking the inner product of the first equation of (1.1) with $|\mathbf{u}|^2\mathbf{u}$, one obtains

$$\begin{aligned}&\frac14\int_{\mathbb{R}^2}|\mathbf{u}|^4dx\Big|_0^t+\int_0^t\int_{\mathbb{R}^2}\Big[\frac12|\nabla|\mathbf{u}|^2|^2+|\nabla\mathbf{u}|^2|\mathbf{u}|^2\Big]dxds\\&=\int_0^t\int_{\mathbb{R}^2}\Big[\mathcal{Q}\mathrm{div}(\mathbf{u}\mathbf{u}_j)+\mathrm{div}(\mathtt{F}-I)+\mathcal{P}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top)\Big]\cdot|\mathbf{u}|^2\mathbf{u}dxds\\&=-\int_0^t\int_{\mathbb{R}^2}\Big[(-\Delta)^{-1}\mathrm{div}\mathrm{div}(\mathbf{u}\otimes\mathbf{u})\mathbf{u}\cdot\nabla|\mathbf{u}|^2+(\mathtt{F}-I):\nabla(|\mathbf{u}|^2\mathbf{u})\\&\quad+(-\Delta)^{-1}\mathrm{curl}\mathrm{div}((\mathtt{F}-I)(\mathtt{F}-I)^\top):\mathrm{curl}(|\mathbf{u}|^2\mathbf{u})\Big]dxds.\end{aligned}\tag{3.8}$$

Since

$$|\nabla(|\mathbf{u}|^2\mathbf{u})|\le M|\nabla\mathbf{u}|\mathbf{u}^2,$$

the right hand side of (3.8) is controlled by

$$\frac12\int_0^t\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2|\mathbf{u}|^2dxds+M\int_0^t g(s)\left(\int_{\mathbb{R}^2}|\mathbf{u}|^4dx\right)ds+M\int_0^t\int_{\mathbb{R}^2}|\mathtt{F}-I|^4dxds$$

with

$$g(s)=1+\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dx\in L^1(0,1).$$

The last term above is bounded by $M\varepsilon_0 B(t)$. Thus, we deduce from (3.8) that

$$\int_{\mathbb{R}^2}|\mathbf{u}|^4dx\le M\Big[\int_0^t g(s)\left(\int_{\mathbb{R}^2}|\mathbf{u}|^4dx\right)ds+\varepsilon_0 B(t)\Big].$$

An easy application of Gronwall's inequality then completes the proof. □

We conclude this section with a result concerning the Holder continuity of $\mathbf{u}$. The the standard notation for Holder norms will be adapted

$$\langle w\rangle^\alpha=\sup_{\substack{x,y\in\mathbb{R}^2\\x\ne y}}\frac{|w(x)-w(y)|}{|x-y|^\alpha}$$

for $\alpha\in(0,1)$.

Lemma 3.4. *For all $t \in (0,1]$, it holds*

$$\langle \mathbf{u}(\cdot,t)\rangle^{\alpha} \le M\left[\left(\varepsilon_0+\int_{\mathbb{R}^2}|\nabla \mathbf{u}|^2dx\right)^{\frac{1-\alpha}{2}}\times\left(\int_{\mathbb{R}^2}|\mathcal{P}\dot{\mathbf{u}}|^2dx\right)^{\frac{\alpha}{2}}+\varepsilon_0^{\frac{1-\alpha}{2}}B(t)^{\frac{\alpha}{2}}\right]$$

for $\alpha \in (0,1)$.

Proof. For any $p>2$, Sobolev's embedding theorem implies

$$\langle \mathbf{u}(\cdot,t)\rangle^{\alpha} \le M\|\nabla \mathbf{u}\|_{L^p(\mathbb{R}^2)}$$

with $\alpha = 1-\frac{2}{p}$. Therefore, it holds

$$\langle \mathbf{u}(\cdot,t)\rangle^{\alpha} \le M\Big[\|\mathfrak{G}\|_{L^p}+\|\mathtt{F}-I\|_{L^p}\Big]. \tag{3.9}$$

Since

$$\|\mathtt{F}-I\|_{L^p} \le M\varepsilon_0^{\frac{1-\alpha}{2}}B(t)^{\frac{\alpha}{2}},$$

and

$$\begin{aligned}\|\mathfrak{G}\|_{L^p} &\le M\left(\int_{\mathbb{R}^2}\mathfrak{G}^2dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^2}|\nabla\mathfrak{G}|^2dx\right)^{\frac{p-2}{2p}}\\ &\le M\left(\varepsilon_0+\int_{\mathbb{R}^2}|\nabla \mathbf{u}|^2dx\right)^{\frac{1-\alpha}{2}}\left(\int_{\mathbb{R}^2}|\mathcal{P}\dot{\mathbf{u}}|^2dx\right)^{\frac{\alpha}{2}}.\end{aligned}$$

Substituting these estimates back in (3.9) gives the desired.

□

4. Pointwise bounds for F and Estimate for $B(T)$

In this section we derive pointwise bounds for the deformation gradient $\mathtt{F}$ in terms of A. First we show that $\mathtt{F}$ remains bounded for large time in terms of $B(1)$ and $A(T)$. Then in the next step, we obtain pointwise bounds for $\mathtt{F}$ near $t=0$; a rather delicate analysis is required here, owing to the degradation of the smoothness estimates for $\mathbf{u}$ near the initial layer. All the assumptions and notations described in the previous section will continue to hold throughout this section.

With aid of (3.5), the pointwise L^∞ bound of $\mathtt{F}-I$ can be stated as

Lemma 4.1. *Under the same assumption as Theorem 1.1, we have*

$$B(T) \le M(\varepsilon_0^{\theta}+A(T)+B(T)^2).$$

Proof. **Step One: $T>1$.** Integrating (3.6) along particle trajectories for $t\in[1,T]$ yields

$$\|\mathtt{F}-I\|_{L^\infty}^4(t) \le \|\mathtt{F}_0-I\|_{L^\infty}^4(1)+M\int_1^T\|\mathfrak{G}\|_{L^\infty}^4ds. \tag{4.1}$$

We estimate the last term here as follows.

$$\begin{aligned}\|\mathfrak{G}\|_{L^\infty}^4 &\le M\|\mathfrak{G}\|_{W^{1,4}}^4\\ &\le M\left[\int_{\mathbb{R}^2}\mathfrak{G}^4dx+\int_{\mathbb{R}^2}|\nabla\mathfrak{G}|^4dx\right]\\ &\le M\int_{\mathbb{R}^2}|\nabla\mathfrak{G}|^2dx\int_{\mathbb{R}^2}\mathfrak{G}^2dx+\int_{\mathbb{R}^2}|\Delta\mathfrak{G}|^2dx\int_{\mathbb{R}^2}|\nabla\mathfrak{G}|^2dx\\ &\le MA(T)\left[\int_{\mathbb{R}^2}|\mathcal{P}\dot{\mathbf{u}}|^2dx+\int_{\mathbb{R}^2}|\nabla\mathcal{P}\dot{\mathbf{u}}|^2dx\right].\end{aligned}$$

Since $t\ge 1$, it follows from above that

$$\begin{aligned}\int_1^T\|\mathfrak{G}\|_{L^\infty}^4ds &\le MA(T)\int_1^T\int_{\mathbb{R}^2}(|\mathcal{P}\dot{\mathbf{u}}|^2+|\nabla\mathcal{P}\dot{\mathbf{u}}|^2)dx\\ &\le M(\varepsilon_0^2+A(T)^2).\end{aligned}$$

Thus, for $T>1$, it holds

$$B(T)\le M\Big[\varepsilon_0+B(1)+A(T)\Big].$$

Step Two: $T\le 1$**.**
Let Γ be the fundamental solution for the Laplace operator

$$\Gamma(x)=\frac{1}{2\pi}\ln|x|.$$

Denoting the element in the reference coordinate by X, and the flow map is given by

$$\frac{d}{dt}x(t,X)=\mathbf{u}(x(t,X),t),\quad\text{with}\quad x(0,X)=X.$$

¿From the definition of $\mathtt{F}$, it holds

$$\partial_{X_i}=\mathtt{F}_{mi}\partial_{x_m}.$$

Since $\Delta\mathcal{G}=\nabla\mathcal{P}\dot{\mathbf{u}}$, along the trajectory, it holds, after changing variables

$$\begin{aligned}\mathfrak{G}\mathtt{F}(x(s),s)&=\Gamma\star\nabla\mathcal{P}\dot{\mathbf{u}}_i(\cdot,s)(x(s,X))\mathtt{F}(x(s,X),s)\\ &\quad+(-\Delta)^{-1}\mathrm{curlcurl}(F-I)(x(s,X),s)\mathtt{F}(x(s,X),s)\\ &\quad+(-\Delta)^{-1}\nabla\mathcal{P}\mathrm{div}((F-I)(F-I)^\top)(x(s,X),s)\mathtt{F}(x(s,X),s)\\ &=\sum_{n=1}^3N_n.\end{aligned}\tag{4.2}$$

Note that the ij entry of N_1 can be written as

$$\begin{aligned}
(N_1)_{ij} &= \Gamma \star \nabla_{x_m} \mathcal{P}\dot{\mathbf{u}}_i(\cdot,s)(x(s,X))\mathtt{F}_{mj}(x(s,X),s) \\
&= \int_{\mathbb{R}^2} \Gamma(x(s,X)-y)\nabla_{y_m}\mathcal{P}\dot{\mathbf{u}}_i(y,s)dy\mathtt{F}_{mj}(x(s,X),s) \\
&= \int_{\mathbb{R}^2} \nabla_{x_m}\Gamma(x(s,X)-y)\mathcal{P}\dot{\mathbf{u}}_i(y,s)dy\mathtt{F}_{mj}(x(s,X),s) \\
&= \int_{\mathbb{R}^2} \nabla_{X_j}\Gamma(x(s,X)-y)\mathcal{P}\dot{\mathbf{u}}_i(y,s)dy \\
&= \frac{d}{ds}\left(\int_{\mathbb{R}^2} \nabla_{X_j}\Gamma(x(s,X)-y)\mathbf{u}_i(y,s)dy\right) \\
&\quad - \int_{\mathbb{R}^2} \Gamma_{mk}(x(s,X)-y)\left(\mathbf{u}_k(x(s,X),s)-\mathbf{u}_k(y,s)\right)\mathbf{u}_i(y,s)dy\mathtt{F}_{mj}(x(s,X),s) \\
&\quad - \int_{\mathbb{R}^2} \Gamma_m(x(s,X)-y)\mathcal{Q}(\mathbf{u}\cdot\nabla\mathbf{u})(y,s)dy\mathtt{F}_{mj}(x(s,X),s) \\
&= \sum_{m=1}^{3} N_{1_m}.
\end{aligned}$$

Observe that, since

$$\nabla_{X_j}\Gamma \star \mathbf{u}_i(\cdot,s) = \Gamma_m \star \mathbf{u}_i(\cdot,s)\mathtt{F}_{mj}(x(s,X),s),$$

one obtains

$$\begin{aligned}
\|\nabla_{X_j}\Gamma \star \mathbf{u}_i(\cdot,s)\|_{L^\infty} &\le M\|\Gamma_m \star \mathbf{u}_i(\cdot,s)\|_{L^\infty} \\
&\le M\Big[\|\mathbf{u}\|_{L^p} + \|\mathbf{u}\|_{L^3}\Big]
\end{aligned} \tag{4.3}$$

for any $p \in [1,2)$ (see for example (1.32) in [9]). From Lemma 3.2, we have

$$\begin{aligned}
\|\mathbf{u}\|_{L^p}^p &\le M\int_{\mathbb{R}^2}(1+|x|^2)^{-\frac{p}{2}}(1+|x|^2)^{\frac{p}{2}}|\mathbf{u}|^p dx \\
&\le M\left(\int_{\mathbb{R}^2}(1+|x|^2)^{-\frac{p}{2-p}}\right)^{1-\frac{p}{2}}\left(\int_{\mathbb{R}^2}(1+|x|^2)|\mathbf{u}|^2dx\right)^{\frac{p}{2}} \\
&\le M\varepsilon_0^{\frac{p}{2}}
\end{aligned}$$

if p is chosen sufficiently close to 2. In addition, since $t \le T \le 1$,

$$\|\mathbf{u}\|_{L^3} \le M\|\mathbf{u}\|_{L^2}^{\frac{1}{3}}\|\mathbf{u}\|_{L^4}^{\frac{2}{3}} \le M\varepsilon_0^{\frac{1}{6}}.$$

Substituting these estimates back into (4.3), we then find that $\Gamma_j \star \mathbf{u}_i(\cdot,s)$ can be bounded as

$$\|\nabla_{X_j}\Gamma \star \mathbf{u}_i(s)\|_{L^\infty} \le M\varepsilon_0^{\theta} \quad \text{for all} \quad 0 \le s \le 1,$$

and hence

$$\left|\int_0^t N_{1_1}ds\right| \le M\varepsilon_0^{\theta}.$$

Let $\phi(y,s)$ be the integrand of N_{1_2}. Then since $|\Gamma_{mk}(x)|\le C$ for $|x|\ge 1$,

$$\int_0^t\int_{|x(s)-y|\ge 1}|\phi(y,s)|dy|\mathtt{F}_{mj}(x(s,X),s)|ds\le M\varepsilon_0.$$

On the other hand, for $\alpha\in(\frac12,1)$,

$$\begin{aligned}
&\int_0^t\int_{|x(s)-y|\le 1}|\phi(y,s)|dy|\mathtt{F}_{mj}(x(s,X),s)|ds\\
&\le M\int_0^t\langle\mathbf{u}(\cdot,s)\rangle^\alpha\int_{|x(s)-y|\le t}|x(s)-y|^{\alpha-2}|\mathbf{u}(y,s)|dyds\\
&\le M\left(\int_0^1 r^{\frac{4(\alpha-2)}{3}}rdr\right)^{\frac34}\sup_{0\le s\le t}\|\mathbf{u}(\cdot,s)\|_{L^4}\int_0^t\langle\mathbf{u}(\cdot,s)\rangle^\alpha ds\\
&\le M\int_0^t\langle\mathbf{u}(\cdot,s)\rangle^\alpha ds
\end{aligned}\tag{4.4}$$

by Lemma 3.3. Applying Lemma 3.4, we then obtain that

$$\begin{aligned}
&\int_0^t\int_{|x(s)-y|\le 1}|\phi(y,s)|dy|\mathtt{F}_{mj}(x(s,X),s)|ds\\
&\le M\varepsilon_0^{\frac{1-\alpha}{2}}B(t)^{\frac{\alpha}{2}}+M\int_0^t\left(\varepsilon_0+\int_{\mathbb{R}^2}|\nabla\mathbf{u}|dx\right)^{\frac{1-\alpha}{2}}\left(\int_{\mathbb{R}^2}|\mathcal{P}\dot{\mathbf{u}}|^2dx\right)^{\frac{\alpha}{2}}ds\\
&\le M(\varepsilon_0^\theta+B(t))+M\left(\int_0^t s^{-\alpha}ds\right)^{\frac12}\left(\varepsilon_0+\int_0^t\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\right)^{\frac{1-\alpha}{2}}\\
&\quad\times\left(\int_0^t\int_{\mathbb{R}^2}s|\mathcal{P}\dot{\mathbf{u}}|^2dxds\right)^{\frac{\alpha}{2}}\\
&\le M(\varepsilon_0^\theta+B(t)+A(t)^{\frac12}).
\end{aligned}$$

For N_{1_3}, from the definition of $\mathcal{Q}$, it holds

$$\begin{aligned}
N_{1_3}&=-\int_{\mathbb{R}^2}\Gamma_m(x(s)-y)\mathcal{Q}(\mathbf{u}\cdot\nabla\mathbf{u})(y,s)dy\mathtt{F}_{mj}(x(s,X),s)\\
&=-\int_{\mathbb{R}^2}\Gamma_m(x(s)-y)\partial_m(-\Delta)^{-1}\mathrm{div}(\mathbf{u}\cdot\nabla\mathbf{u})(y,s)dy\mathtt{F}_{mj}(x(s,X),s)\\
&=\int_{\mathbb{R}^2}\Gamma(x(s)-y)\mathrm{div}(\mathbf{u}\cdot\nabla\mathbf{u})(y,s)dy\mathtt{F}_{mj}(x(s,X),s)\\
&=\int_{\mathbb{R}^2}\Gamma_{im}(x(s)-y)\mathbf{u}_i(y,s)\mathbf{u}_m(y,s)dy\mathtt{F}_{mj}(x(s,X),s)\\
&=\int_{\mathbb{R}^2}\Gamma_{im}(x(s)-y)(\mathbf{u}_i(y,s)-\mathbf{u}_i(x(s),s))\mathbf{u}_m(y,s)dy\mathtt{F}_{mj}(x(s,X),s),
\end{aligned}$$

since

$$\mathbf{u}_i(x(s),s)\int_{\mathbb{R}^2}\Gamma_{im}(x(s)-y)\mathbf{u}_m(y,s)dy=0$$

due to $\mathrm{div}\mathbf{u} = 0$. Hence, one can proceed as the argument for N_{1_2} to obtain

$$\left| \int_0^t N_{1_3} ds \right| \leq M(\varepsilon_0^\theta + B(t) + A(t)^{\frac{1}{2}}).$$

For N_2, one has

$$\begin{aligned}
\left| \int_0^t N_2 ds \right| &\leq \int_0^t \|(-\Delta)^{-1}\mathrm{curlcurl}(F - I)(x(s), s)\|_{L^\infty} ds \\
&\leq M \int_0^t \|\mathtt{F} - I\|_{L^2}^{\frac{1}{2}} \|\mathrm{curlcurl}(F - I)(x(s), s)\|_{L^2}^{\frac{1}{2}} ds \\
&\leq M A(t)^{\frac{1}{4}} \int_0^t \|\nabla \mathcal{P} \dot{\mathbf{u}}\|_{L^2}^{\frac{1}{2}} ds \\
&\leq M A(t)^{\frac{1}{4}} \left(\int_0^t s^{-\frac{2}{3}} ds \right)^{\frac{3}{4}} \left(\int_0^t \|s \nabla \mathcal{P} \dot{\mathbf{u}}\|_{L^2}^2 ds \right)^{\frac{1}{4}} \\
&\leq M A(t)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, one can bound N_3 as

$$\left| \int_0^t N_3 ds \right| \leq M A(t)^{\frac{1}{2}}.$$

Combining all these estimates, we then obtain that

$$\left| \int_0^t \mathfrak{G}\mathtt{F}(x(s), s) ds \right| \leq M(\varepsilon_0^\theta + B(t) + A(t)^{\frac{1}{2}}). \tag{4.5}$$

On the other hand, from the second equation of (1.1) and the definition of $\mathfrak{G}$, one has

$$\frac{d}{ds}(\mathtt{F}(x(s), s) - I) + (\mathtt{F}(x(s), s) - I)\mathtt{F}(x(s), s) = \mathfrak{G}\mathtt{F}(x(s), s).$$

Integrating this identity along the trajectory and using (4.5), it follows

$$\left| (\mathtt{F}(x(t), t) - I) \Big|_{s=0}^{s=t} \right| \leq M \int_0^t |\mathtt{F} - I| ds + M(\varepsilon_0^\theta + B(t) + A(t)^{\frac{1}{2}}).$$

This further implies

$$|\mathtt{F}(x(t), t) - I| \leq M \int_0^t |\mathtt{F} - I| ds + M(\varepsilon_0^\theta + B(t) + A(t)^{\frac{1}{2}}).$$

Applying Gronwall's inequality gives,

$$\sup_{0 \leq t \leq T} \|\mathtt{F} - I\|_{L^\infty}^2 \leq M \left[\varepsilon_0^\theta + A(T) + B(T)^2 \right],$$

as required. $\square$

5. Proof of Theorem 1.1

In this section we apply the *a priori* estimates of Section 2 and Section 3 to complete the proof of Theorem 1.1 stated in Introduction.

To begin with, we consider an approximating system to (1.1)

$$\begin{cases}\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}+\delta(-\Delta)^2\mathbf{u}+\nabla P=\mathrm{div}(\mathtt{F}\mathtt{F}^\top),\\ \partial_t\mathtt{F}+\mathbf{u}\cdot\nabla\mathtt{F}=\nabla\mathbf{u}\mathtt{F},\\ \mathrm{div}\mathbf{u}=0,\\ (\mathbf{u}(x,0),\mathtt{F}(x,0))=(\mathbf{u}_0,\mathtt{F}_0),\end{cases}\tag{5.1}$$

where $\delta>0$ is the parameter. Note that constraints (1.2) and (1.3) still hold true since their verification only involves the equation of $\mathtt{F}$. Thanks to the higher order diffusion term, $\delta(-\Delta)^2\mathbf{u}$, the flow map is smooth and the trajectory is well defined for initial data $(\mathbf{u}_0,\mathtt{F}_0)\in L^2(\mathbb{R}^2)$. Moreover the global existence of solutions to (5.1) can be established through a standard energy method.

Let $(\mathbf{u}_0,\mathtt{F}_0)$ be initial data as described in Theorem 1.1, and let $(\mathbf{u}^n,\mathtt{F}^n)$ be the solution of (5.1) with $\delta=\frac{1}{n}$. The *a priori* estimates as in Lemma 2.2 and Lemma 3.1 then can be obtained as

$$A(T)\le M\Big(\varepsilon_0^\theta+A(T)^2+B(T)^2\Big)$$

and

$$B(T)\le M\Big(\varepsilon_0^\theta+A(T)+B(T)^2\Big)$$

for all $T<\infty$, where A and B are now as in (1.6) and (1.7), but with $(\mathbf{u},\mathtt{F})$ being replaced by $(\mathbf{u}^n,\mathtt{F}^n)$. Using the fact that A and B are continuous in t and the hypothesis that ε_0 is small, we may then conclude that

$$A(T)+B(T)\le C\varepsilon_0^\theta,$$

for some universal positive constant C.

Based on this bound, up to a subsequence, we can assume that for an arbitrary $T>0$

$$\mathbf{u}^n\to\mathbf{u}\quad\text{weak}^*\text{ in}\quad L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^2))$$

and

$$\mathtt{F}^n\to\mathtt{F}\quad\text{weak}^*\text{ in}\quad L^2(0,T;L^2(\mathbb{R}^2))\cap L^\infty((0,T)\times\mathbb{R}^2).$$

Then it is a routine argument to get (see for example [22]) that as $n\to\infty$

$$\frac{1}{n}\Delta\mathbf{u}^n\to 0\quad\text{in}\quad\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2)$$

$$\mathbf{u}^n\cdot\nabla\mathbf{u}^n\to\mathbf{u}\cdot\nabla\mathbf{u}\quad\text{in}\quad\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2)$$

and

$$\mathtt{F}^n\otimes\mathbf{u}^n-\mathbf{u}^n\otimes\mathtt{F}^n\to\mathtt{F}\otimes\mathbf{u}-\mathbf{u}\otimes\mathtt{F}\quad\text{in}\quad\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2).$$

Taking the limit as $n\to\infty$ in the momentum equation of (1.1), one has

$$\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\Delta\mathbf{u}+\nabla P=\mathrm{div}(\overline{\mathtt{F}\mathtt{F}^\top-I})\quad\text{in}\quad\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^2),\tag{5.2}$$

where the notation $\overline{f}$ means the weak limit in L^2 of $\{f^n\}$.

It is only left to show the strong convergence of the deformation gradient in L^2.

Lemma 5.1. $\mathtt{F}^n-I\to\mathtt{F}-I$ *converges strongly in* $L^2(\mathbb{R}^2)$.

Proof. Multiplying the second equation in (1.1) by $(\mathtt{F}^n - I)$, we have, using $\mathrm{div}(\mathtt{F}^n)^\top = 0$

$$\frac{1}{2}\int_{\mathbb{R}^2} |\mathtt{F}^n(t) - I|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} |\mathtt{F}_0^n - I|^2 dx = \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u}^n : (\mathtt{F}^n(\mathtt{F}^n)^\top - I) dx ds; \tag{5.3}$$

and similarly

$$\frac{1}{2}\int_{\mathbb{R}^2} |\mathtt{F}(t) - I|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} |\mathtt{F}_0 - I|^2 dx = \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : (\mathtt{F}\mathtt{F}^\top - I) dx ds; \tag{5.4}$$

On the other hand, taking the inner product of the momentum equation with $\mathbf{u}^n$ gives

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u}^n : (\mathtt{F}^n(\mathtt{F}^n)^\top - I) dx ds \\ &\quad = -\frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{u}^n(t)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{u}_0^n|^2 dx - \int_0^t \int_{\mathbb{R}^2} |\nabla \mathbf{u}^n|^2 dx ds; \end{aligned} \tag{5.5}$$

while the inner product of (5.2) with $\mathbf{u}$ gives

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : \overline{\mathtt{F}\mathtt{F}^\top - I} dx ds \\ &\quad = -\frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{u}(t)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{u}_0|^2 dx - \int_0^t \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx ds. \end{aligned} \tag{5.6}$$

Due to the convexity of $x \mapsto x^2$, it can be deduced from (5.5) and (5.6) that

$$\limsup_{n\to\infty} \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u}^n : (\mathtt{F}^n(\mathtt{F}^n)^\top - I) dx ds \le \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : \overline{\mathtt{F}\mathtt{F}^\top - I} dx ds,$$

and hence this in turn, according to (5.3)-(5.4) and the strong convergence of $\mathtt{F}_0^n$, implies that

$$\frac{1}{2}\int_{\mathbb{R}^2} \overline{|\mathtt{F}(t) - I|^2} dx \le \frac{1}{2}\int_{\mathbb{R}^2} |\mathtt{F} - I|^2 dx + \mathfrak{R} \tag{5.7}$$

with

$$\mathfrak{R} = \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : \overline{\mathtt{F}\mathtt{F}^\top - I} dx ds - \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : (\mathtt{F}\mathtt{F}^\top - I) dx ds.$$

Next we claim that

$$\mathfrak{R} = 0.$$

Indeed, observing that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : \overline{\mathtt{F}\mathtt{F}^\top - I} dx ds \\ &\quad = \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : (\mathtt{F}^n(\mathtt{F}^n)^\top - I) dx ds \\ &\quad = \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : (\mathtt{F}^n - I + (\mathtt{F}^n - I)(\mathtt{F}^n - I)^\top) dx ds \\ &\quad = \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : (\mathtt{F} - I) dx ds + \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : (\mathtt{F}^n - I)(\mathtt{F}^n - I)^\top dx ds \\ &\quad = \int_0^t \int_{\mathbb{R}^2} \nabla \mathbf{u} : (\mathtt{F} - I) dx ds + \frac{1}{2}\lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right] : (\mathtt{F}^n - I)(\mathtt{F}^n - I)^\top dx ds. \end{aligned} \tag{5.8}$$

Since $\nabla\mathbf{u}+(\nabla\mathbf{u})^\top \in L^2(\mathbb{R}^2)$ is symmetric, there exists a complex-valued matrix $\mathcal{S} \in L^4(\mathbb{R}^2)$ such that

$$\nabla\mathbf{u} + (\nabla\mathbf{u})^\top = \mathcal{S}\mathcal{S}^\top.$$

Thus the last term in (5.8) takes the form

$$\frac{1}{2}\lim_{n\to\infty}\int_0^t\int_{\mathbb{R}^2}|\mathcal{S}^\top(\mathtt{F}^n - I)|^2 dxds,$$

which is bigger than

$$\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}|\mathcal{S}^\top(\mathtt{F} - I)|^2 dxds.$$

Therefore, (5.8) further implies that

$$\int_0^t\int_{\mathbb{R}^2}\nabla\mathbf{u} : \overline{\mathtt{F}\mathtt{F}^\top - I}dxds \geq \int_0^t\int_{\mathbb{R}^2}\nabla\mathbf{u} : (\mathtt{F}\mathtt{F}^\top - I)dxds, \tag{5.9}$$

and it follows

$$\mathfrak{R} \geq 0.$$

A similar argument as from (5.8) to (5.9) with $\nabla\mathbf{u}$ being replaced by $-\nabla\mathbf{u}$ gives

$$\int_0^t\int_{\mathbb{R}^2}\nabla\mathbf{u} : \overline{\mathtt{F}\mathtt{F}^\top - I}dxds \leq \int_0^t\int_{\mathbb{R}^2}\nabla\mathbf{u} : (\mathtt{F}\mathtt{F}^\top - I)dxds,$$

and it follows

$$\mathfrak{R} \leq 0.$$

Combining those two inequalities together gives the desired claim.

(5.7) now implies that

$$\frac{1}{2}\int_{\mathbb{R}^2}\overline{|\mathtt{F}(t) - I|^2}dx \leq \frac{1}{2}\int_{\mathbb{R}^2}|\mathtt{F} - I|^2 dx.$$

Since $\overline{|\mathtt{F}(t) - I|^2} \geq |\mathtt{F} - I|^2$ almost everywhere because of the convexity of the map $x \mapsto x^2$, it follows that

$$\frac{1}{2}\int_{\mathbb{R}^2}\overline{|\mathtt{F}(t) - I|^2}dx = \frac{1}{2}\int_{\mathbb{R}^2}|\mathtt{F} - I|^2 dx.$$

This, combining with the weak convergence of $\mathtt{F}^n - I$ in $L^2(\mathbb{R}^2)$, implies $\mathtt{F}^n(t) - I$ converges to $\mathtt{F}(t) - I$ strongly in $L^2(\mathbb{R}^2)$ for almost all $t > 0$. □

With the strong convergence of $\mathtt{F}^n - I$ in L^2 and its uniform bound in L^∞, constraints (1.2) and (1.3) are preserved for the limit function $\mathtt{F}$.

References

[1] Y. Chen, P. Zhang: *The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions.* Comm. Partial Differential Equations 31 (2006), 1793-1810.

[2] J.-Y. Chemin, N. Masmoudi, *About lifespan of regular solutions of equations related to viscoelastic fluids.* SIAM J. Math. Anal. 33 (2001), 84-112 .

[3] R. Coifman, P. L. Lions, Y. Meyer, S. Semmes: *Compensated compactness and Hardy spaces.* J. Math. Pures Appl. (9) 72 (1993), 247-286.

[4] P. Constantin, *Complex fluids and Lagrangian particles.* Topics in mathematical fluid mechanics, 1-21, Lecture Notes in Math., 2073, Springer, Heidelberg, 2013.

[5] P. Constantin, *Remarks on complex fluid models.* Mathematical aspects of fluid mechanics. 70-87, London Math. Soc. Lecture Note Ser., 402, Cambridge Univ. Press, Cambridge, 2012.

[6] P. Constantin, M. Kliegl, *Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress.* Arch. Ration. Mech. Anal. 206 (2012), 725-740.

[7] P. Constantin, G. Seregin, *Holder continuity of solutions of 2D Navier-Stokes equations with singular forcing.* Nonlinear partial differential equations and related topics, 87-95, Amer. Math. Soc. Transl. Ser. 2, 229, Amer. Math. Soc., Providence, RI, 2010.

[8] P. Constantin, W. Sun, *Remarks on Oldroyd-B and related complex fluid models.* Commun. Math. Sci. 10 (2012), 33-73.

[9] D. Hoff: *Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data.* J. Differential Equations 120 (1995), 215-254.

[10] X. Hu, D. Wang, *Global existence for the multi-dimensional compressible viscoelastic flows*. J. Differential Equations 250 (2011) 1200-1231.

[11] R. Kupferman, C. Mangoubi, E. Titi, *A Beale-Kato-Majda breakdown criterion for an Oldroyd-B fluid in the creeping flow regime*. Commun. Math. Sci. 6 (2008), 235-256.

[12] Z. Lei, C. Liu, Y. Zhou: *Global solutions for incompressible viscoelastic fluids*. Arch. Ration. Mech. Anal. 188 (2008), 371–398.

[13] Z. Lei, T. C. Sideris, Y. Zhou, em Almost global existence for 2-D incompressible isotropic elastodynamics. To appear in Trans. Am. Math. Soc.

[14] F. H. Lin, C. Liu, P. Zhang: *On hydrodynamics of viscoelastic fluids*. Comm. Pure Appl. Math. 58 (2005), 1437–1471.

[15] F. H. Lin, P. Zhang: *On the initial-boundary value problem of the incompressible viscoelastic fluid system*. Comm. Pure Appl. Math. 61 (2008), 539–558.

[16] CPAM INITIAL BOUNDARY

[17] P. L. Lions, N. Masmoudi: *Global solutions for some Oldroyd models of non-Newtonian flows*. Chinese Ann. Math. Ser. B 21 (2000), 131-146.

[18] N. Masmoudi: *Global existence of weak solutions to the FENE dumbbell model of polymeric flows.* Invent. Math. 191 (2013), 427-500.

[19] J. Qian, Z. Zhang, *Global well-posedness for compressible viscoelastic fluids near equilibrium.* Arch. Ration. Mech. Anal. 198 (2010), 835-868.

[20] T. C. Sideris, B. Thomases, *Global existence for three-dimensional incompressible isotropic elastodynamics via the incompressible limit*. Comm. Pure Appl. Math. 58 (2005), 750-788.

[21] T. C. Sideris, B. Thomases, *Global existence for three-dimensional incompressible isotropic elastodynamics*. Comm. Pure Appl. Math. 60 (2007), 1707-1730.

[22] R. Temam, *Navier-Stokes equations. Theory and numerical analysis.* Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[23] B. Thomases, M. Shelley, *Emergence of singular structures in Oldroyd-B fluids*. Phys. Fluids 19, 103 (2007).

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